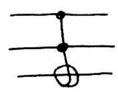
Lecture 12 2/25/10 Universal set of fault-tolerant gate gadgets

① Fault-tolerant Toffoli [Shor, 9605011]

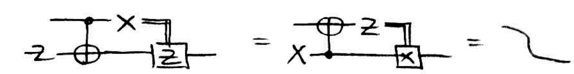


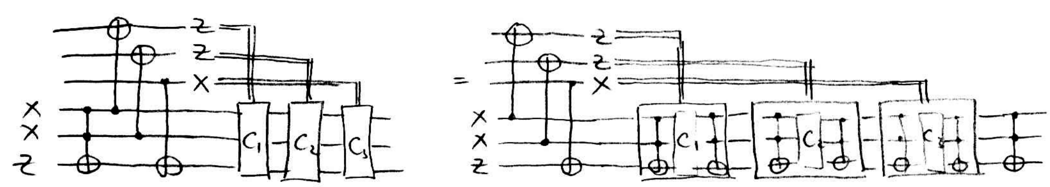
$T|a,b,c\rangle = |a, b, c \oplus ab\rangle$

- construction using computation + teleportation (using adaptive classical control)
  1. prepare the ancilla $\sum_{ab} T|a,b,0\rangle = \sum_{a,b} |a,b,a\cdot b\rangle =$ 

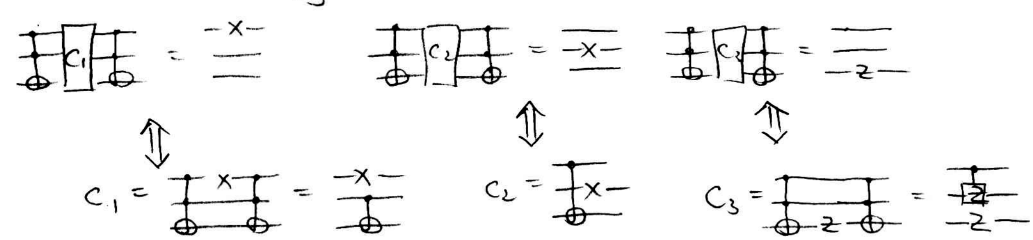
  2. teleport into it

  one-qubit teleportation:





thus corrections should satisfy



000	+	100	+
001	−	101	−
010	+	110	−
011	−	111	+

$\Rightarrow$ corrections are CSS Cliffords, can be applied fault tolerantly

To prepare the encoded ancilla fault tolerantly:

note $|+\rangle_1|+\rangle_2|+\rangle_3|+\rangle_4 = |+\rangle \otimes (|A\rangle + |B\rangle)$

$|A\rangle = \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |111\rangle)$

$|B\rangle = X_3|A\rangle$

$\xrightarrow[\Lambda_1(Z_4)]{\Lambda_1(\Lambda_2(Z_3))}$ $|0\rangle \otimes (|A\rangle + |B\rangle)_{234}$ $+ |1\rangle \otimes \underbrace{\Lambda_2(Z_3) Z_4 (|A\rangle + |B\rangle)_{234}}_{|A\rangle - |B\rangle}$ $= |+\rangle \otimes |A\rangle + |-\rangle \otimes |B\rangle$

$\hookrightarrow$ measure & correct repeatedly on

assume self-dual CSS code with $\bar{X} = X^{\otimes n}$, $\bar{Z} = Z^{\otimes n}$

$\Rightarrow \overline{\Lambda(Z)}, \bar{Z}$ are both transversal

$\Rightarrow$ encode qubits 2, 3, 4 into the code

use a verified cat state $|0^n\rangle + |1^n\rangle$ for the first qubit

note: $\Lambda_1(\Lambda_2(Z_3))$ uses a Toffoli

A similar technique will work for any C_3 gate, eg., $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$.
The corresponding state can be teleported into, requiring only Clifford corrections.
The ancilla is an eigenstate of a Clifford gate, can measure the eigenvalue using controlled Cliffords from a verified cat state.

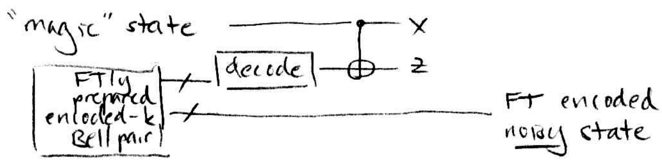
② Magic states distillation [Knill Laflamme Zurek 9610011, Bravyi Kitaev 0403025]

so far: Clifford gates + prep. of certain nonstabilizer states $\Rightarrow$ universality (with classical control)

universality at level 0 $\Rightarrow$ FT universality at level 1

$\Rightarrow \cdots \Rightarrow$ FT universality at level k

<u>new idea</u>: Clifford gates + prep. of certain <u>noisy</u> nonstabilizer states $\Rightarrow$ universality



Advantage: Through the highest level of encoding, all operations are (CSS) Cliffords
$\rightarrow$ easy to simulate (with Pauli noise channels)
$\rightarrow$ one less exRec to count malignant sets within, simplified analysis
Disadvantage: Typically higher overhead (necessarily?)

Note: If Cliffords + noisy magic state prep. $\underset{c.c.}{\Rightarrow}$ universality,
then $\Rightarrow$ prep. of noiseless magic states

$\therefore$ Suffices to study <u>distillation problem</u>:

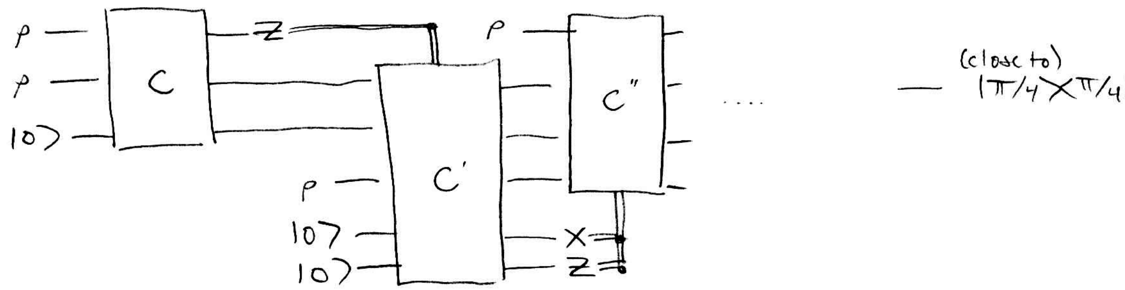
From which noisy states can we asymptotically prepare a magic state,

eg. $|A\rangle = \sum_{a,b=0}^{1} |a, b, a \cdot b\rangle$

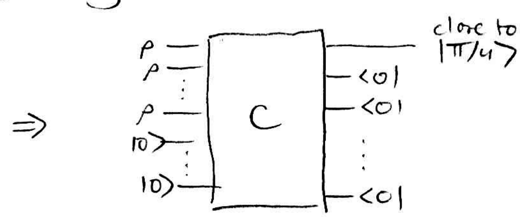
or $|\pi/4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$

using adaptive classical control of stabilizer operations?

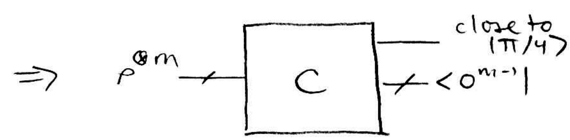
<u>Model</u>. Can repeatedly prepare identical copies of ρ, a single-qubit state.



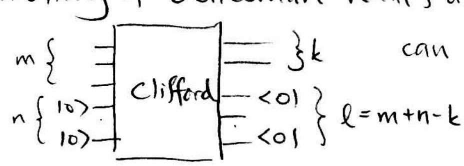
Claim 1: May assume all measurements are postselected +1.



Claim 2: May assume no stabilizer preparations used.



(Corollary of Gottesman-Knill, any

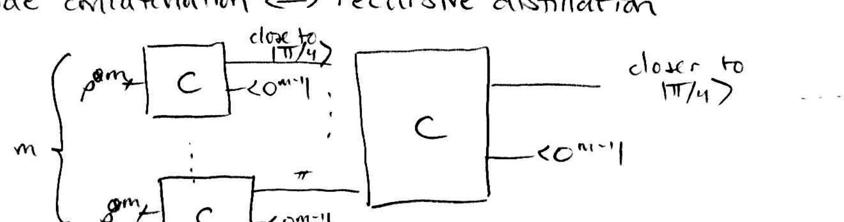


can be implemented with $\max\{0, k-m\}$ extra ancilla qubits!)

$\Rightarrow$ C' encodes into a stabilizer code

stabilizer codes $\Leftrightarrow$ distillation schemes

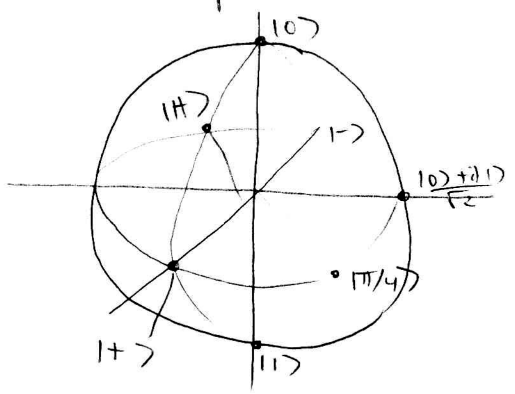
code concatenation $\Leftrightarrow$ recursive distillation



Good codes can be found by trial and error (no general theory).

Note: Need overhead to be polynomial, preferably polylogarithmic.

Bloch sphere



1. symmetrize $\rho = \rho(x,y,z) = \frac{1}{2}(I + xX + yY + zZ)$

about $x=z$ axis by applying I or H with 50/50 prob. at <u>random</u>

$$\Rightarrow \rho' = \frac{1}{2}\left(\rho + H\rho H\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}(1 + xX + yY + zZ) + \frac{1}{2}(1 + xZ - yY + zX)\right)$$

$$= \frac{1}{2}\left(1 + \frac{x+z}{2}X + \frac{x+z}{2}Z\right)$$

$$= \rho\left(\frac{x+z}{2}, 0, \frac{x+z}{2}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1+\frac{x+z}{2} & \frac{x+z}{2} \\ \frac{x+z}{2} & 1-\frac{x+z}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+x' & x' \\ x' & 1-x' \end{pmatrix}$$

2. Decode the code

$$\Rightarrow \rho_{out} = \begin{pmatrix} \langle\bar{0}|\rho^{\otimes m}|\bar{0}\rangle & \langle\bar{0}|\rho^{\otimes m}|\bar{1}\rangle \\ \langle\bar{1}|\rho^{\otimes m}|\bar{0}\rangle & \langle\bar{1}|\rho^{\otimes m}|\bar{1}\rangle \end{pmatrix} \Big/ \text{trace}$$

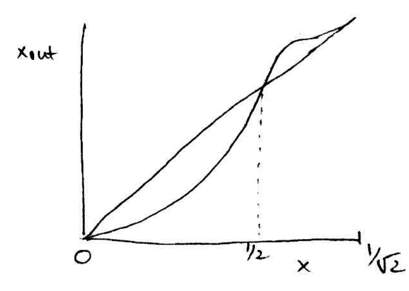
Eg. for the Steane [[7,1,3]] code,

$$|\bar{0}\rangle = \frac{1}{\sqrt{8}}\Big(|0^7\rangle + |0001111\rangle + |0110011\rangle + |1010101\rangle + |0111100\rangle + |1011010\rangle + |1100110\rangle + |1101001\rangle\Big)$$

$$\Rightarrow \langle\bar{0}|\rho^{\otimes m}|\bar{0}\rangle = \frac{1}{8}\left(\rho_{00}^7 + 7\rho_{00}^3(\rho_{01}^4 + \rho_{10}^4) + 7\rho_{00}^3\rho_{11}^4 + 7\cdot 6\,\rho_{00}\rho_{01}^4\rho_{11}^2\right)$$

similarly for the other terms, since $|\bar{1}\rangle = X^{\otimes 7}|\bar{0}\rangle$

$$\Rightarrow \rho_{out} = \rho(x_{out}, 0, x_{out}), \quad x_{out} = \frac{x^3(7+8x^4)}{1+14x^4}$$



<u>Note</u>: No ρ inside octahedron can be distilled, convex combination of stabilizer states.

⋮

Application to $\nu = 5/2$ anyon FQHE.

Open problems: Distillable region, asymptotic distillation rate, n qubit states?

(Tight) tolerable noise threshold upper bounds

based on Gottesman-Knill theorem, magic states distillation

Virmani Huelga Plenio 0408076

Buhrman, Cleve, Laurent, Linden, Schrijver, Unger 0?? R ?

Howard, van Dam 0907.3189

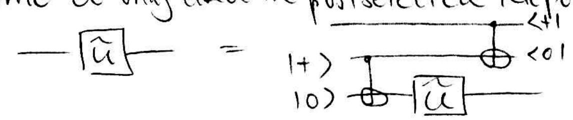
Model: Assume perfect classical control of perfect stabilizer operations (Clifford gates, prep, meas.), and a noisy non-Clifford gate.

Example: $\underset{\text{randomize qubit w/prob. } 4p/3}{\mathcal{E}_{depolarizing}(p)} \circ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}}_{\mathcal{U}}$

Claim: If $p \geq \frac{3}{4} \cdot \frac{6-2\sqrt{2}}{7} \approx 34\%$, then computation is classically simulatable.

Proof:

1. May assume $\tilde{\mathcal{U}}$ only used in postselected teleportation



2. If $\underbrace{(I \otimes \tilde{\mathcal{U}})(|00\rangle + |11\rangle)}_{\text{Jamiolkowski isomorphism of } \tilde{\mathcal{U}}}$ is a (known) mixture of stabilizer states,

then Gottesman-Knill gives efficient classical simulation.

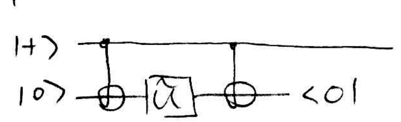
$$= \left(1 - \tfrac{4}{3}p\right)\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & e^{-i\pi/4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\pi/4} & 0 & 0 & 1 \end{pmatrix} + \frac{4}{3}p \cdot \frac{1}{4}\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$\underset{p = \frac{3}{4}\frac{6-2\sqrt{2}}{7}}{=} \left(\frac{4+\sqrt{2}}{14}\right) J(I) + \left(\frac{4+\sqrt{2}}{14}\right) J\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} + \left(\frac{3-\sqrt{2}}{14}\right) J(X) + \left(\frac{3-\sqrt{2}}{14}\right) J(Y) . \checkmark$$

$\square$

Claim: If $p < \frac{3}{4}\frac{6-2\sqrt{2}}{7}$, then the model allows for efficient qu. universality

Proof: Applying



gives $\frac{1}{2}\begin{pmatrix} 1 & e^{-i\pi/4}\frac{3-4p}{3-2p} \\ e^{i\pi/4}\frac{3-4p}{3-2p} & 1 \end{pmatrix}$

$= \rho(x, x, 0)$, $x = \frac{1}{\sqrt{2}} \cdot \frac{3-4p}{3-2p}$

$= \frac{1}{2}$

at $p = \frac{3}{4}\frac{6-2\sqrt{2}}{7}$ $\checkmark$ $\square$