Classical threshold results:

Models:

Formulas: 

Circuits: 

Independent gate failures

Von Neumann '56 : MAJ_3 w/ 0.0073

Pippenger '88: $\varepsilon \geq \frac{1}{2} - \frac{1}{2k}$ for fan-in $\leq k$ gates not sufficient for universal, bounded-error computation

Feder '89 : ditto for circuits

Evans & Schulman '99 : $\frac{1}{2} - \frac{1}{2\sqrt{k}}$ ditto

" " '03: $\beta_k = \frac{1}{2} - \frac{2^{k-2}}{k\binom{k-1}{\frac{k}{2}-\frac{1}{2}}}$ tight for k odd

Hajek & Weller '91 : $k=3$ case ($\beta_3 = 1/6$)

fan-in 2 for formulas

Evans & Pippenger '98 : for NAND gates $\frac{3-\sqrt{7}}{4}$ tight

Unger '07 : tight for all fan-in 2 gates

Quantum FT history (through 2005)

Codes: Shor 9-qubit '95

Steane 7-qubit '95

Calderbank & Shor '96, Steane '96: CSS codes

Gottesman '96, Calderbank, Rains, Shor, Sloane '96: stabilizer codes

Knill, Laflamme '97: general QECC

Fault tolerance:

Shor '96: $\frac{1}{\mathrm{poly}(\log N)}$ error tolerable to simulate N gates

Aharonov & Ben-Or '97, Kitaev '97, Knill Laflamme Zurek '97

constant thresholds based on code concatenation

proofs for $d \geq 5$

all-unitary model, estimated threshold $\sim 10^{-6}$ noise per gate

Gottesman & Preskill '98: claimed proof for $d=3$

"dark ages"

, now standard criterion, improved proof in Aliferis '07

2005: R, Aliferis Gottesman Preskill: proofs for $d=3$,

explicit numerical lower bounds 1.4×10^{-5} & 2.7×10^{-5}

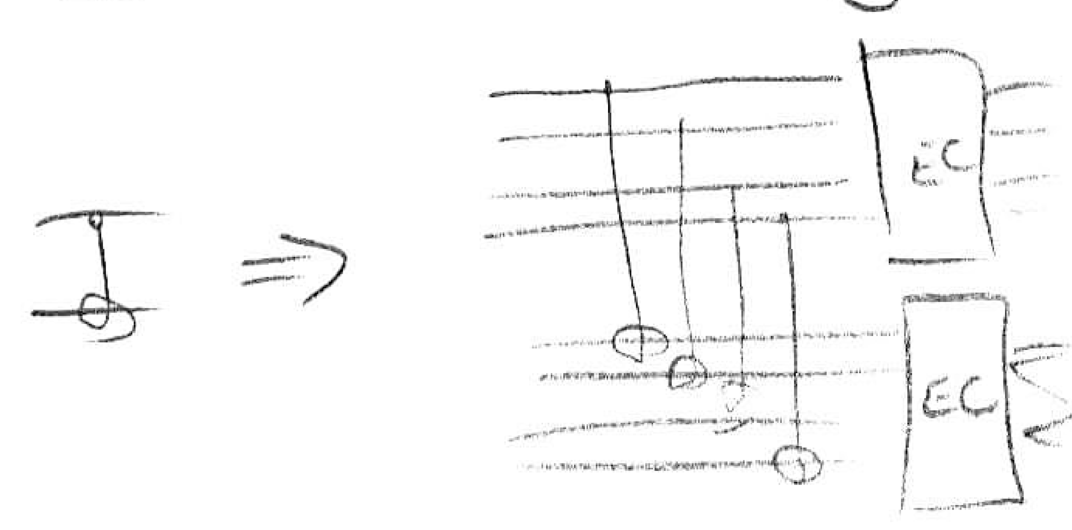
> Steane '02: many simulations, est. 3×10^{-3}

R '04, Knill '04: postselection ...

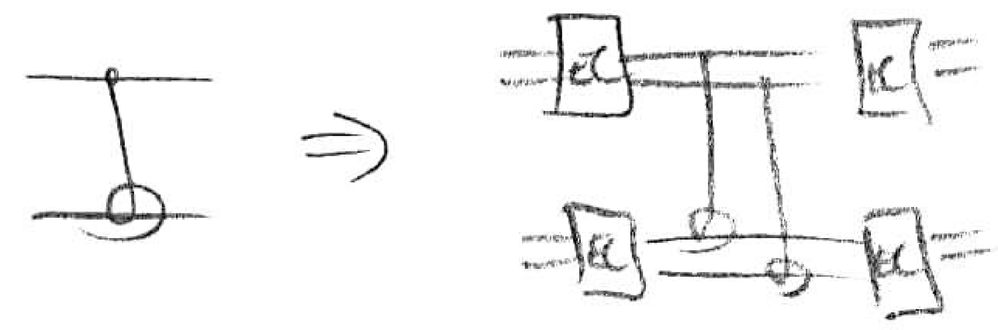
Magic states distillation; Bravyi Kitaev '04, Knill '04

Aharonov & Ben-Or's threshold proof:

"Def": A code block is good if it has ≤ 1 bad subblocks.



$\mathbb{P}$[output blocks good | input blocks good] $= O(p^2)$ if $d \geq 7$



" if $d \geq 5$

not for $d=3$

→ inefficient

→ conservative logical error rate $p \mapsto \binom{m}{2}p^2$ not $\binom{m}{3}p^3$

→ no good for $d=3$

Idea: KLZ 97, R'04, AGP'04: Most blocks should have no bad subblocks

Good

$\underbrace{IXXXX}_{X}$ $\underbrace{IIIII}_{I}$

$\underbrace{\underbrace{IXXXX}_{X}\ \underbrace{IIXXX}_{X}\ \underbrace{X^5}_{X}\ \underbrace{X^5}_{X}\ \underbrace{X^5}_{X}}_{X}$

Bad

$\underbrace{IIXXX}_{X}$

$\underbrace{\underbrace{IIIII}_{I}\ \underbrace{IIXXX}_{X}\ \underbrace{X^5}_{X}\ \underbrace{X^5}_{X}\ \underbrace{X^5}_{X}}_{X}$

Problem: Errors are ambiguous $|00\rangle + |11\rangle$ vs. $|01\rangle + |10\rangle$

Solution: Track errors from their introduction

Block error states. Ideal recursive decoding

<u>Def</u>. A level k-block is k-good if it has at most one level k-1 subblock either in relative error or not k-1 good.

Every bit (0-block) is 0-good.

12/21/2009 A simple threshold proof: (in a style like AGP05 but simplified similar to Aliferis's thesis)

Error model: Adversarial stochastic noise.

Assume that any set of k space-time locations fails with probability $\leq \varepsilon^k$, for $k \geq 1$. Of course such a distribution is dominated by one in which every location fails independently with probability ε.

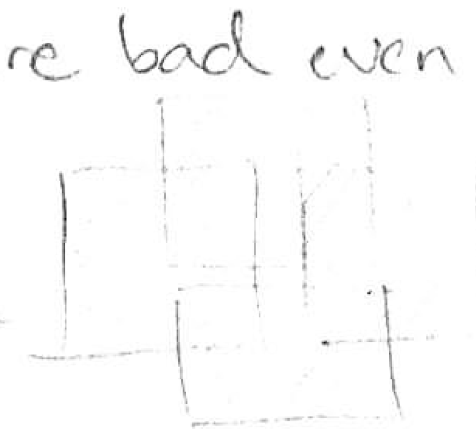
The adversary is given the set of failed locations across all spacetime. (Yes this is unphysical.) He can use randomness to decide on a set of channels, one for each failed location. Every qubit has a corresponding finite-dimensional bath, and to a failed location the adversary can apply an arbitrary channel to the corresponding qubits and baths. The adversary chooses the baths' initial state.

This is a strong noise model, strong enough to be conserved under <u>level reduction</u>. For weaker noise, eg depolarizing noise, a threshold can be determined by using level reduction to reduce to this noise model. <u>Unitary noise</u> can also be accommodated, but will be discussed separately.

Thm 2 concatenation

exRec failures:

Define a level-k exRec to be bad if it contains $> t$ independently bad $(k-1)$-exRecs, where t is the number of correctable errors for the code. Here, two or more k-exRecs are independently bad if they are bad even after assigning shared k-ECs to the later exRecs. For example,



for these four exRecs to be independently bad, the four shaded regions must themselves each be bad.

By induction, then, the probability of r k-exRecs being independently bad is at most $(\varepsilon^{(k)})^r$, where $\varepsilon^{(k)}$ is any upper bound on the probability of a single k-exRec (possibly missing some trailing ECs) being bad. We can easily solve this recursion

$$\varepsilon^{(k)} = A \cdot (\varepsilon^{(k-1)})^{t+1}, \text{ where } A = \binom{\ell}{t+1}$$

and ℓ is an upper bound on the number of $(k-1)$-Recs in any k-exRec.

→

Theorem. There exists a threshold $\varepsilon_t > 0$ such that if $\varepsilon < \varepsilon_t$ then an ideal quantum circuit with T (CSS Clifford) gates, width N and depth D can be simulated by an ε-noisy quantum circuit with

$$T \cdot (\log T)^{O(1)} \text{ gates}, \quad D(\log T)^{O(1)} \text{ depth}, \quad N(\log T)^{O(1)} \text{ width}.$$

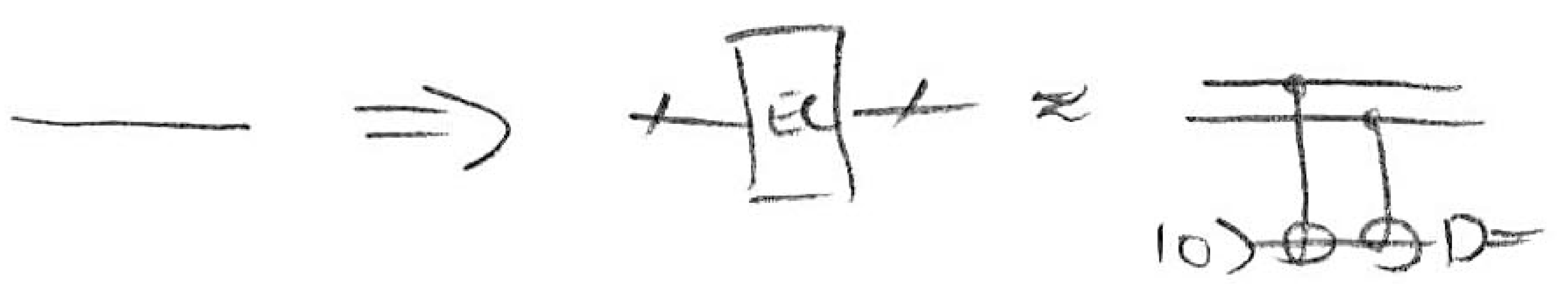
In particular, arbitrarily long computations are possible with polynomial (in fact polylogarithmic) overhead provided the noise is below the threshold.

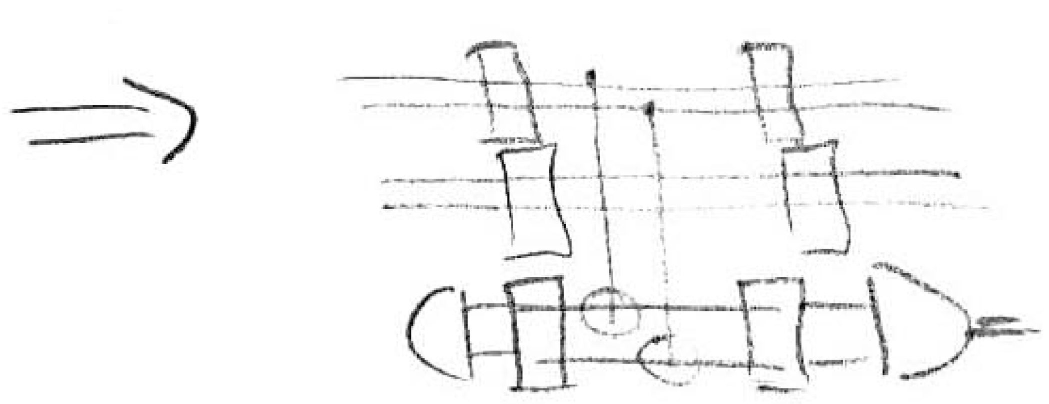
Concatenation.

Compile a circuit using FT gadgets.

Then compile that one, etc.

→ sequence of codes, level $k=0$ qubits $k=1$ one level of concatenation, ...





etc

Remark: Concatenated decoding may be suboptimal.

Claim: The concatenation of two distance-d codes has distance d^2.

Eg. $[\![7,1,3]\!] \longrightarrow [\![49,1,9]\!]$

↓

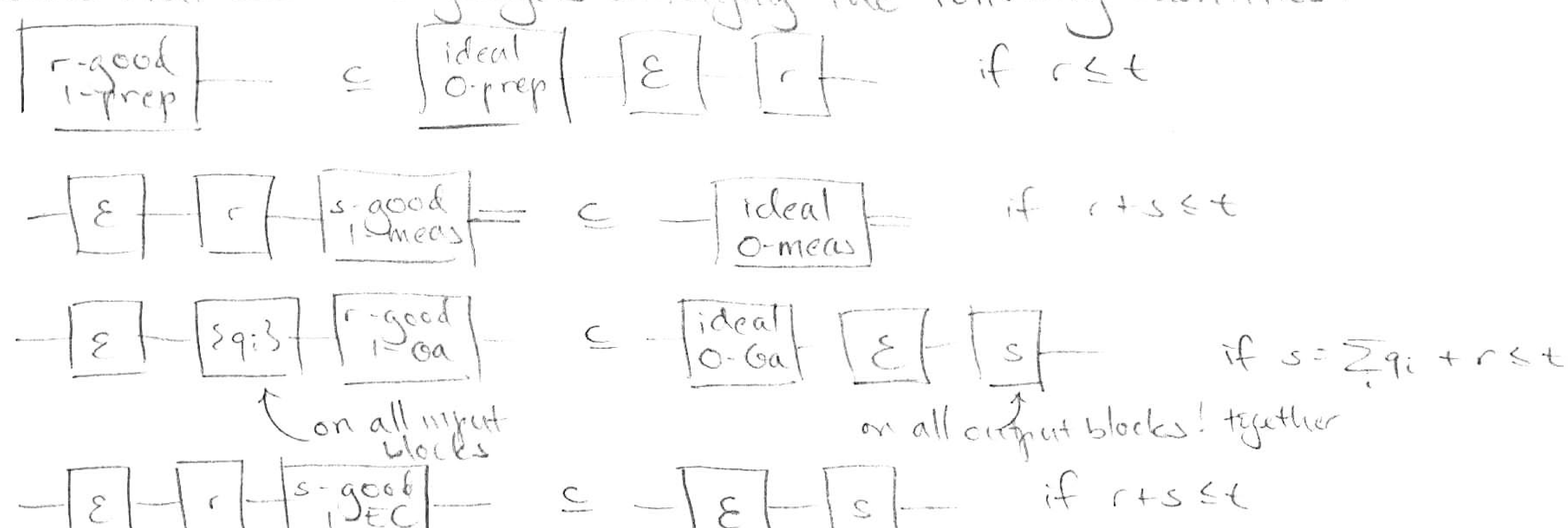
incorrect decoding recursively

(But on random errors, performance should be similar.)

Now let us study the behavior of the circuit for a fixed set of failed locations.

Notation: By $-\boxed{r}-$ applied to a level-1 block of qubits in a circuit diagram, we indicate allowing the adversary to choose r qubits from the block and apply an arbitrary channel to all r qubits and their baths.

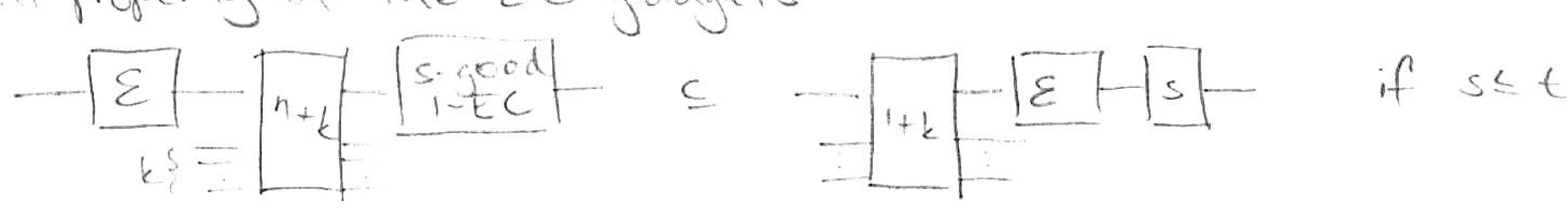
Assume that we have gadgets satisfying the following identities:



The meaning of each identity is that we can take a circuit diagram and replace a local piece of it with the right-hand side, and only potentially increase the adversary's power. $\mathcal{E}$ is the ideal encoder.

If every 1-exRec is good, the above rules allow us to create encoders and pull them all the way across the circuit, leaving just an ideal circuit.

We need to understand what happens in bad exRecs. A bad exRec, on j blocks each of n qubits, can be replaced by $\boxed{jn}$, an arbitrary channel on those jn qubits and baths. To pull encoders past, we need one further important property of the EC gadgets:



Here for the error on the unencoded wire, the adversary is allowed to act on all the baths for the qubits in the associated code block.

Now we can pull the encoders all the way through, leaving behind a circuit with certain failed locations where there had been independently bad exRecs. Rinse, repeat. ✓

Four remaining issues are to justify the above rules, particularly for the ECs, to study "malignant" sets of locations, to extend the noise model, and to compute explicit thresholds.