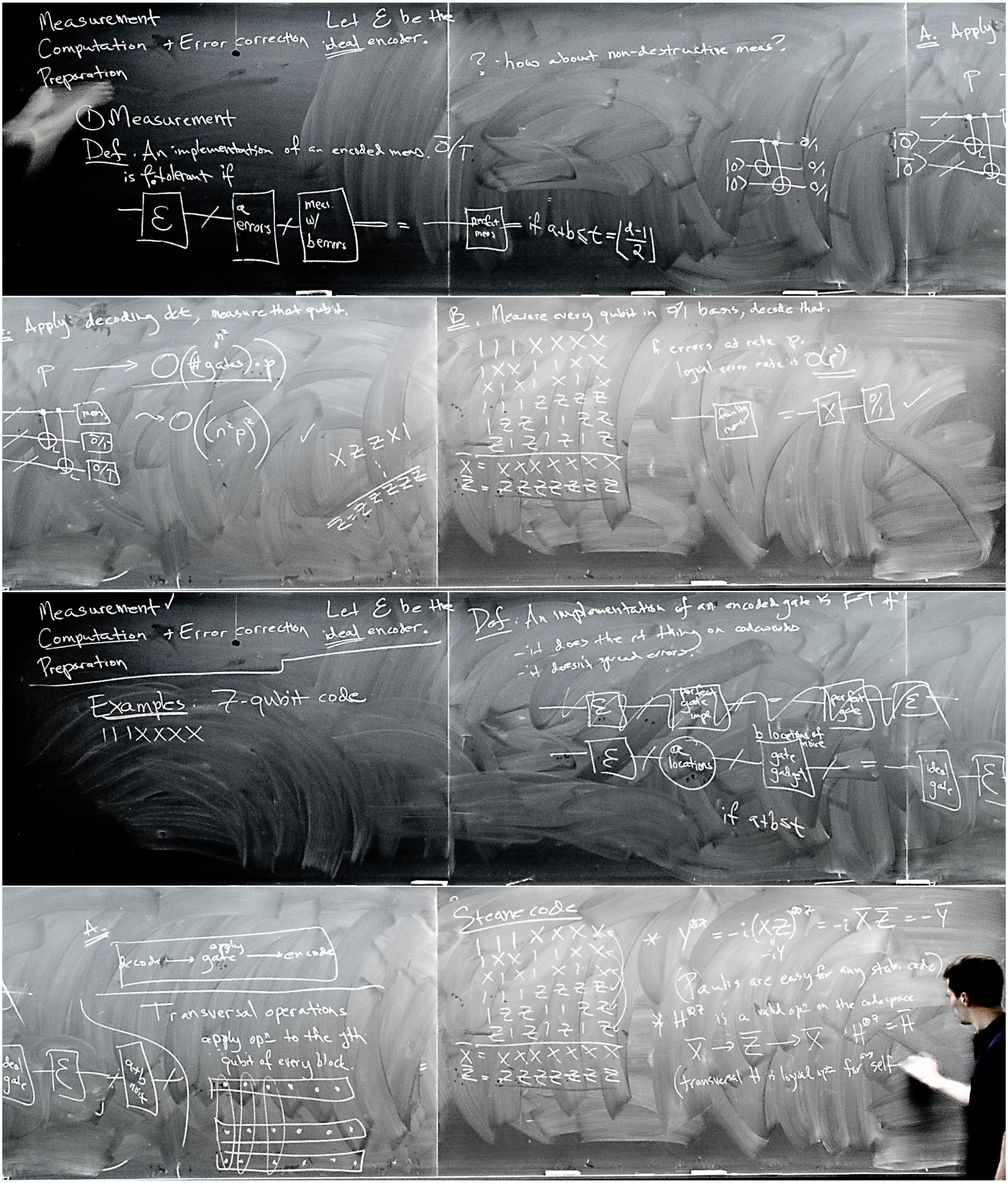
CS 798: Quantum Fault Tolerance lecture 6: Fault-tolerant measurement and computation (1/28/10)



Measurement ✓

Computation + Error correction

Preparation

Let $\mathcal{E}$ be the ideal encoder.

XXXXXX

ZZZZZZ

$\bar{X}_1 =$ X X I I I I

$\bar{Z}_1 =$ I Z I I I Z

$\bar{X}_2 =$ X I X I I I

$\bar{Z}_2 =$ I I Z I I Z

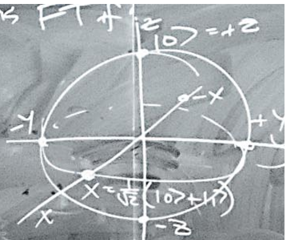
⋮

$X^{\otimes 6} \to Y^{\otimes 6}$

$= -(X^{\otimes 6})(Z^{\otimes 6})$

Def: An implementation of an encoded gate is FT if:

- it does the rt thing on codewords
- it doesn't spread errors.



— $\mathcal{E}$ — a locations — b locs / gate gadget — = — ideal gate — $\mathcal{E}$

if $a+b \le t$

$|0\rangle = +Z$

$T: X \to Y \to Z \to X$

$* \; T^{\otimes 7}$

$111XXXX \to 111YYYY \in S$

$= (111XXXX)$

$\cdot (111ZZZZ)$

$111ZZZZ \to 111XXXX$

$\bar{X} \to -\bar{Y} \to \bar{Z} \to \bar{X}$

$\bar{H}\,\bar{T}^{+}\bar{H}$

— ideal gate — $\mathcal{E}$ — a+b noisy —

(works for any doubly even self-dual CSS code)

Steane code

1 1 1 X X X X

1 X X 1 1 X X

X 1 X 1 X 1 X

1 1 1 Z Z Z Z

1 Z Z 1 1 Z Z

Z 1 Z 1 Z 1 Z

$\bar{X} =$ X X X X X X X

$\bar{Z} =$ Z Z Z Z Z Z Z

$* \; Y^{\otimes 7} = -i(XZ)^{\otimes 7} = -i\bar{X}\bar{Z} = -\bar{Y}$, with $XZ = -iY$

(Paulis are easy for any stab. code)

$* \; H^{\otimes 7}$ is a valid op² on the codespace

$\bar{X} \to \bar{Z} \to \bar{X} \qquad H^{\otimes 7} = \bar{H}$

(transversal H is logical op² for any self-dual CSS code)

Measurement ✓

Computation + Error correction

Preparation

Let $\mathcal{E}$ be the ideal encoder.

Def: An implementation of an encoded gate is

- it does the rt thing on codewords
- it doesn't spread errors.

$\Rightarrow$ any single-qubit Clifford ✓

$*$ encoded CNOT ✓ $\Rightarrow$ logical CNOT

$(111XXXX) \otimes (1111111)$

$\xrightarrow{\text{CNOT}} (111XXXX) \otimes (111XXXX) \in S$ ✓

$(1111111) \otimes (111XXXX)$

$(X) \otimes (I) \to (X) \otimes (X)$ ✓

$(I) \otimes (X) \to (I) \otimes (X)$ ✓

$Z^{\otimes 6} \to X^{\otimes 6}$ ✓

$X^{\otimes 6} \to Y^{\otimes 6}$ ✗

$= -(X^{\otimes 6})(Z^{\otimes 6})$

$\bar{X}_1 = X^{\otimes 7} \otimes 1^{\otimes 7}$

$\to X^{\otimes 7} \otimes X^{\otimes 7} = \bar{X}_1 \bar{X}_2$

$\bar{Z}_1 \to \bar{Z}_1$

$\bar{Z}_2 \to \bar{Z}_1 \otimes \bar{Z}_2$

$\bar{X}_2 \to \bar{X}_2$

stuff

$\bar{X}_1 =$ X X X X X X X 1 1 1 1 1 1 1

$\bar{Z}_1 =$ Z ⋯ Z 1 ⋯ 1

$\bar{X}_2 =$ 1 1 ⋯ 1 X ⋯ X

$\bar{Z}_2 =$ 1 1 ⋯ 1 Z ⋯ Z

Steane code

1 1 1 X X X X

1 X X 1 1 X X

X 1 X 1 X 1 X

1 1 1 Z Z Z Z

1 Z Z 1 1 Z Z

Z 1 Z 1 Z 1 Z

$\bar{X} =$ X X X X X X X

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$\bar{X} \to \bar{Z} \to \bar{X} \qquad H^{\otimes 7} = \bar{H}$

(transversal H is logical op² for any self-dual CSS code)

CSS code)

$$\begin{array}{cccccccc}
X&X&X&X&1&1&1&1\\
Z&Z&Z&Z&1&1&1&1\\
1&1&1&1&X&X&X&X\\
1&1&1&1&Z&Z&Z&Z
\end{array}$$

$\bar{X}_{1,1} = XX11 \longrightarrow \bar{X}_{11}\cdot\bar{X}_{21}$

$\bar{X}_{12} = X1X1 \longrightarrow \bar{X}_{12}\cdot\bar{X}_{22}$

$\bar{Z}_{11} = 1Z1Z \longrightarrow \bar{Z}_{11}$

$\bar{Z}_{12} = 11ZZ \longrightarrow \bar{Z}_{12}$

$\bar{X}_{21}$: $XX11 \longrightarrow \bar{X}_{21}$

$\bar{X}_{22}$: $X1X1 \longrightarrow \bar{X}_{22}$

$\bar{Z}_{21}$: $1Z1Z \longrightarrow \bar{Z}_{11}\bar{Z}_{21}$

$\bar{Z}_{22}$: $11ZZ \longrightarrow \bar{Z}_{12}\bar{Z}_{22}$

Steane code

$$\begin{array}{ccccccc}
1&1&1&X&X&X&X\\
1&X&X&1&1&X&X\\
X&1&X&1&X&1&X\\
1&1&1&Z&Z&Z&Z\\
1&Z&Z&1&1&Z&Z\\
Z&1&Z&1&Z&1&Z
\end{array}$$

$\bar{X} = XXXXXXX$

$\bar{Z} = ZZZZZZZ$

* $Y^{\otimes 7} = -i(\underset{-iY}{XZ})^{\otimes 7} = -i\bar{X}\bar{Z} = -\bar{Y}$

(Paulis are easy for any stab. code)

* $H^{\otimes 7}$ is a valid op. on the codespace

$\bar{X} \to \bar{Z} \to \bar{X}$ $\quad H^{\otimes 7} = \bar{H}$

(transversal H is logical op. for any self-dual CSS code)

SS code)

FT gate gadget

$-\mathcal{E}-/-\boxed{a_1}-/-\boxed{\text{gate } b}-/- \; = \; -\boxed{\text{ideal gate}}-\mathcal{E}-/- \; a_1+a_2+b$

$-\mathcal{E}-/-\boxed{a_2}-/-$ $\quad$ if $a_1+a_2+b \le t$

[Chen, Chung, Cross, Zeng, Chuang '08]: no stabilizer (subsystem) code has a computational universal set of transversal gates. ~allow permutation

[Eastin, Knill '08]: no arbitrary QECC has a transversal gate set giving "strict" universality. ~not allow permutation

[9702029]

Def: Strict universality

A set S of gates is strictly universal if $\exists\, n_0$ s.t. on any $n \ge n_0$ qubits S generates a dense subgroup of $SU(2^n)$.

Def: S is computationally universal if it can be used to simulate to within ε error any n-qubit unitary using t gates from a strictly universal set with space overhead polylog$(n, t, \frac{1}{\varepsilon})$

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SS code)

5-qubit code

$$\begin{array}{ccccc}
X&Z&Z&X&1\\
1&X&Z&Z&X\\
X&1&X&Z&Z\\
Z&X&1&X&Z
\end{array}$$

$\bar{X} = XX\cdots X$

$\bar{Z} = ZZ\cdots Z$

* transversal CNOT?

$XZZX1 \otimes 11111$

$\longrightarrow XZZX1 \otimes X11X1 \notin S$

* transversal T: $X \to Y \to Z \to X$

$XZZX1 \longrightarrow YXXY1 = (X1XZZ)\cdot(ZX1X\ldots$ $\in S$

$\bar{X} \to Y^{\otimes 5} \to \bar{Z}$ ($= \bar{Y}$)

FT gate gadget

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Gottesman [9702029]

Any stabilizer code

$X111 \longrightarrow XXX1$

$Z111 \longrightarrow ZZZ1$

+ cyclic permutations.