CS 798: Quantum Fault Tolerance lecture 5: Threshold 1/2 for erasure error recap

Preskill Ch. 7 lecture notes
Gottesman 0904.2557

Theorem: Threshold erasure noise is 1/2.

Proof ideas:

(1) Large random codes give good protection against random errors.

Theorem. For any $\varepsilon > 0$, $\exists\, n_0$ large enough s.t. for all $n \geq n_0$ $\exists$ a (CSS, stabilizer) QECC $\mathcal{C}$ encodes 1 qubit such that

Remarks:

* Generalizes to Pauli noise

Theorem: Consider a single-qubit Pauli channel $\mathcal{P}$ that applies X, Y or Z w/ respective prob's p_X, p_Y, p_Z. Let $p_I = 1 - p_X - p_Y - p_Z$.
Let $H = H_2(\{p_I, p_X, p_Y, p_Z\}) = -p_I \log_2 p_I - \cdots - p_Z \log_2 p_Z$.
Then for any $R < 1 - H$, $\exists$ a n-qubit QECC that encodes Rn qubits, such that the prob. of incorrectly decoding n copies of the channel is $e^{-\Omega(n)}$.

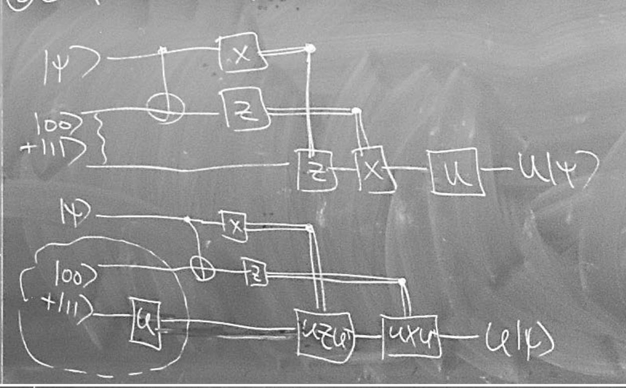
Remarks:

* Generalizes to Pauli noise

Theorem: Consider a single-qubit Pauli channel $\mathcal{P}$ that applies X, Y or Z w/ respective prob's p_X, p_Y, p_Z. Let $p_I = 1 - p_X - p_Y - p_Z$.
Let $H = H_2(\{p_I, p_X, p_Y, p_Z\}) = -p_I \log_2 p_I - \cdots - p_Z \log_2 p_Z$.
Then for any $R < 1 - H$, $\exists$ a n-qubit QECC that encodes Rn qubits, such that the prob. of incorrectly decoding n copies of the channel is $e^{-\Omega(n)}$.

* Best distance of any n-qubit QECC is $\leq \lfloor n/3 \rfloor$ (Rains's bound)
* Encoding into a size-n stabilizer code takes $O(n^2)$ local Clifford gates.
[Aaronson & Gottesman '04]: $O\left(\frac{n^2}{\log n}\right)$

(2) Computation + Teleportation



Preskill Ch. 7 lecture notes
Gottesman 0904.2557

- U Clifford operator
  $U = H$ $\quad HXH = Z$, $HZH = X$
- U is not a Clifford

Theorem: The Clifford group together with the "$\pi/8$ gate" $P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \propto e^{i\frac{\pi}{8} Z}$ is a universal gate set

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $U = P$

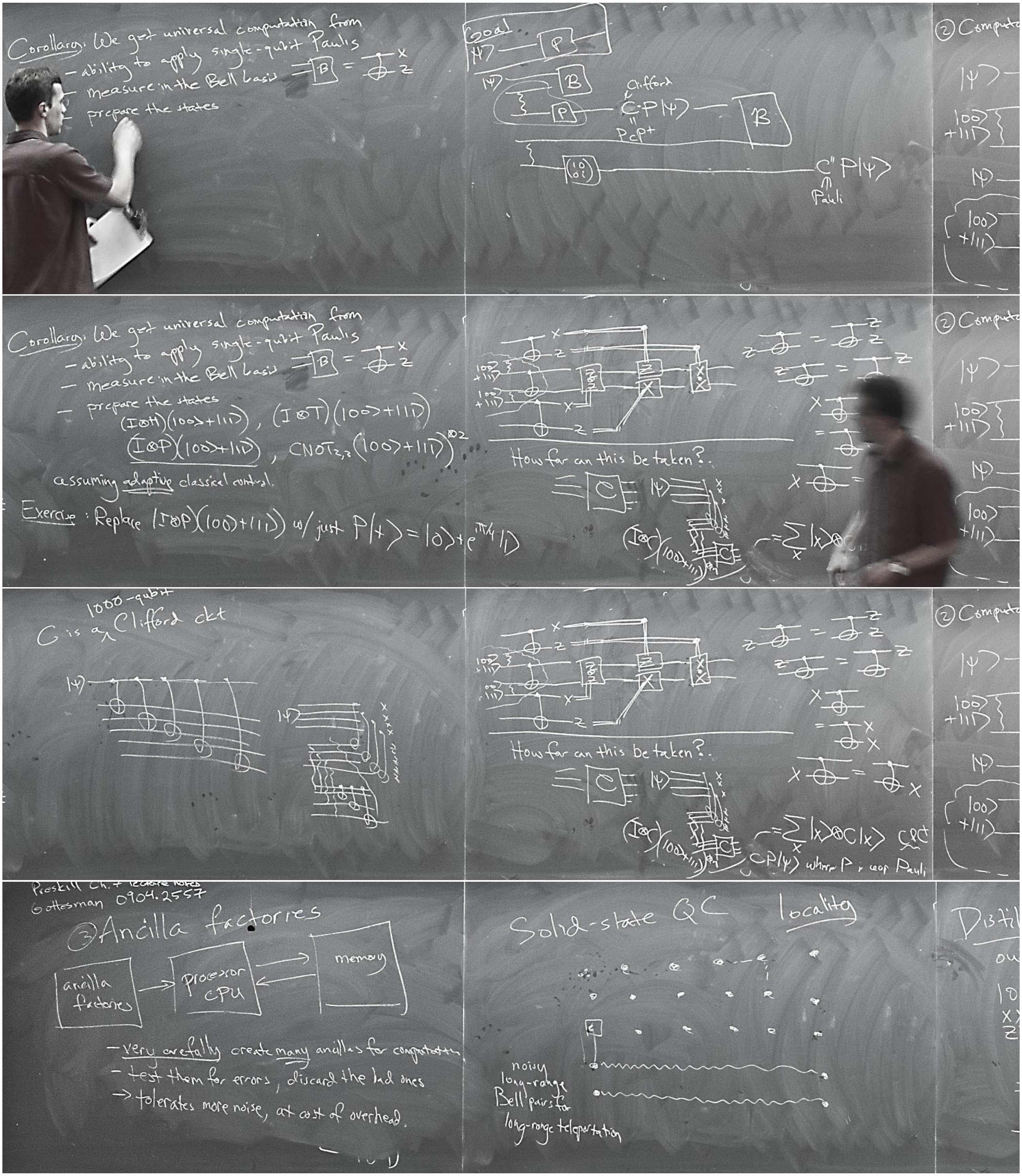
$$P Z P^\dagger = P P^\dagger Z = Z$$

$$P X P^\dagger = e^{-i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} X$$

$\rightarrow$ note $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \in$ Clifford

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} Z \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = Z$$

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} X \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \pm Y$$


Corollary: We get universal computation from
single-qubit Paulis
- ability to apply
- measure in the Bell basis
- prepare the states
Goal
Clifford
Pauli
(I⊗H)(|00>+|11>), (I⊗T)(|00>+|11>)
(I⊗P)(|00>+|11>), CNOT2,3(|00>+|11>)⊗2
assuming adaptive classical control.
Exercise: Replace (I⊗P)(|00>+|11>) w/ just P|+> = |0> + e^iπ/4 |1>
How far can this be taken?
C is a 1000-qubit Clifford ckt
CP|ψ> where P is a Pauli
② Compute
Preskill Ch. 7 lecture notes
Gottesman 0904.2557
③ Ancilla factories
ancilla factories
processor CPU
memory
- very carefully create many ancillas for computation
- test them for errors, discard the bad ones
→ tolerates more noise, at cost of overhead.
Solid-state QC
locality
noisy long-range Bell pairs for long-range teleportation
Distil

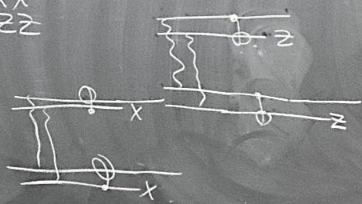
Distillation: take many noisy copies of a state

output one less noisy copy.

$|00\rangle + |11\rangle$

XX
ZZ

XX ZZ   XX ZZ CX_{13} CX_{24} XX ZZ   XX ZZ



$|01\rangle + |10\rangle$:
XX
−ZZ

$s_1 s_3$ X X 1 1
s_2 Z Z 1 1
s_3 1 1 X X
s_4 1 1 Z Z

meas Z_3 ⟹
$s_1 s_3$ X X 1 1
s_2 Z Z 1 1
t_1 1 1 Z 1   with $t_1 \in_R \{\pm 1\}$ known
$s_2 s_4$ 1 1 Z Z

meas Z_4 ⟹
$s_1 s_3$ X X 1 1
s_2 Z Z 1 1
t_1 1 1 Z 1   with $t_2 = t_1 s_2 s_4$ known
t_2 1 1 1 Z

$\mathbb{P}(++) = 1 - p$

$\mathbb{P}(+-) = \mathbb{P}(-+) = \mathbb{P}(--) = p/3$

$s_2 = -1 \iff$ X or Y error on first pair

$s_4 = -1 \iff$ " " second pair

...up Pauli

Answer

start w/ a smaller code

$\to$ error rate $< \frac{1}{\log^2 T} \leq \frac{1}{t}$

$m = \log t = \log\log T$ qubits

$O(m^2)$ overhead $= O((\log\log T)^2)$

The overhead

Goal: Simulate an ideal QC with T gates.

∴ need an effective error rate $\lesssim \frac{1}{T}$. $\mathbb{P}[\text{logical erasure}] = e^{-\Omega(n)}$

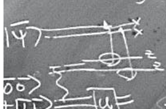
$\implies n \sim \log T$

$\implies$ encoding ckt uses $\sim n^2 = \log^2 T$ gates

$\implies$ overhead is $2^{\log^2 T}$

④ Logical teleportation

−Claim: transversal Bell meas's allow implementing a logical Bell measurement for any stab. code.



X Z Z X 1
1 X Z Z X
X 1 X Z Z
Z X 1 X Z

X Z Z X 1
1 X Z Z X
X 1 X Z Z
Z X 1 X Z
X X X X X X
Z Z Z Z Z Z Z

$\bar{X} = XXXXX$
$\bar{Z} = ZZZZZ$

Answer

start w/ a smaller code

$\to$ error rate $< \frac{1}{\log^2 T} \leq \frac{1}{t}$

$m = \log t = \log\log T$ qubits

$O(m^2)$ overhead $= O((\log\log T)^2)$

The ov
Goal

④ Logical teleportation

−Claim: transversal Bell meas's allow implementing a logical Bell measurement for any stab. code.

X Z Z X 1 X 1 1 X 1
1 X Z Z X 1 X 1 1 X
X 1 X Z Z X 1 X 1 1
Z X 1 X Z 1 X 1 X 1
1 Z Z 1 1 X Z Z X 1
1 1 Z Z 1 1 X Z Z X
1 1 1 Z Z X 1 X Z Z
Z 1 1 1 Z Z X 1 X Z
1 1 1 1 1 X X X X X
Z Z Z Z Z Z Z Z Z Z

$\sim$

X 1 1 X 1 | 1 Z Z 1 1
1 X 1 1 X | 1 1 Z Z 1
X 1 X 1 1 | 1 1 1 Z Z
1 X 1 X 1 | Z 1 1 1 Z

$\bar{X} = XXXXX$
$\bar{Z} = ZZZZZ$

The ov
Goal

④ Logical teleportation

Claim: transversal Bell meas's allow implementing a logical Bell measurement for any stab. code

X Z Z X 1 X 1 1 X 1
1 X Z Z X 1 X 1 1 X
X 1 X Z Z X 1 X 1 1
Z X 1 X Z 1 X 1 X 1
1 Z Z 1 1 X Z Z X 1
1 1 Z Z 1 1 X Z Z X
1 1 Z Z X 1 X Z Z
Z 1 1 1 Z Z X 1 X Z
1 1 1 1 1 X X X X X
Z Z Z Z Z Z Z Z Z Z

$\bar{X}$ = X X X X X X

$\bar{Z}$ = Z Z Z Z Z Z

X 1 1 X 1 | 1 Z Z 1 1
1 X 1 1 X | 1 1 Z Z 1
X 1 X 1 1 | 1 1 1 Z Z
1 X 1 X 1 | Z 1 1 1 Z

$\xrightarrow[\text{meas } X_5]{\text{meas } X_1}$

unchanged

t_1 X 1 1 1 1
t_2 1 X 1 1 1
t_3 1 1 X 1 1
t_4 1 1 1 X 1
t_5 1 1 1 1 X
1 1 1 1 1 X X X X X

The

Goal

Z 1 1
Z Z 1
1 Z Z
1 1 Z

$\xrightarrow[\text{meas } X_5]{\text{meas } X_1}$

X 1 1 X 1 1 Z Z 1 1
1 X 1 1 X 1 1 Z Z 1
X 1 X 1 1 1 1 1 Z Z
1 X 1 X 1 Z 1 1 1 Z

t_1 X 1 1 1 1
t_2 1 X 1 1 1
t_3 1 1 X 1 1
t_4 1 1 1 X 1
t_5 1 1 1 1 X
u_1 1 1 1 1 1 Z 1 1 1 1

$\bar{X}$ = X X X X X $\qquad \bar{X} = t_1 t_2 \cdots t_5 \cdots X_{11}$

$\bar{Z}$ =

u_j = outcome of meas Z_{5+j}

$u_1 \in_R \{\pm 1\}$ $\quad u_1 = (-1)^{\tilde{u}_1}$ $\quad \tilde{u}_1 \in_R \{0,1\}$

$\tilde{u}_1 + \tilde{u}_2 + \tilde{t}_3 + \tilde{t}_5 = 0 \mod 2$

$\tilde{u}_1 + \tilde{u}_3 + \tilde{t}_1 + \tilde{t}_3 + \tilde{t}_4 + \tilde{t}_5 = 0$

$\tilde{u}_1 + \tilde{u}_4 + \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4 = 0$

$\tilde{u}_1 + \tilde{u}_5 + \tilde{t}_2 + \tilde{t}_4 = 0$