Logic and Computation

CS245

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Propositional Logic

Agenda

- Syntax
- Semantics
- Tautological Consequence
- Adequate Sets
- Hilbert System Proofs

Semantics

Informally, *semantics* of a logic describe how to interpret formulas. A *set* is a collection of objects called *members* or *elements*.

In propositional logic, we need to give *meaning* to atoms, connectives, and formulas.

Semantics (informally)

Let A and B be two formulas that express propositions $\mathcal{A}$ and $\mathcal{B}$. Intuitively, we give the following meanings :

$\neg A$	Not $\mathcal{A}$
$A \wedge B$	$\mathcal{A}$ and $\mathcal{B}$
$A \vee B$	$\mathcal{A}$ or $\mathcal{B}$
$A \Rightarrow B$	If $\mathcal{A}$ then $\mathcal{B}$
$A \Leftrightarrow B$	$\mathcal{A}$ iff $\mathcal{B}$

Semantics

Formally, semantics is a function that mapps a formula to a value in $\{0, 1\}$ (also known as *truth table*).

A	$\neg A$
1	0
0	1

Semantics

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

A *truth valuation* is a function with the set of all proposition symbols as domain and $\{0, 1\}$ as range.

Formula Values

The *value* assigned to formulas by a truth valuation t is defined by recursion:

[1] $p^t \in \{0, 1\}$.

[2] $(\neg A)^t = \begin{cases} 1 & \text{if } A^t = 0 \\ 0 & \text{if } A^t = 1 \end{cases}$

[3] $(A \wedge B)^t = \begin{cases} 1 & \text{if } A^t = B^t = 1 \\ 0 & \text{otherwise} \end{cases}$

[3] $(A \vee B)^t = \begin{cases} 1 & \text{if } A^t = 1 \text{ or } B^t = 1 \\ 0 & \text{otherwise} \end{cases}$

[4] $(A \Rightarrow B)^t = \begin{cases} 1 & \text{if } A^t = 0 \text{ or } B^t = 1 \\ 0 & \text{otherwise} \end{cases}$

[5] $(A \Leftrightarrow B)^t = \begin{cases} 1 & \text{if } A^t = B^t \\ 0 & \text{otherwise} \end{cases}$

Formula Values (Example)

Suppose $A = p \vee q \Rightarrow q \wedge r$.

- If $p^t = q^t = r^t = 1$, then $A^t = 1$. (why?)
- If $p^{t_1} = q^{t_1} = r^{t_1} = 0$, then $A^{t_1} = 1$. (why?)

Theorem. For any $A \in Form(\mathcal{L}^p)$ and any truth valuation, $A^t \in \{0, 1\}$.

Satisfiability

Let Σ denote a set of formulas and

$$\Sigma^t = \begin{cases} 1 & \text{if for each } B \in \Sigma, B^t = 1 \\ 0 & \text{otherwise} \end{cases}$$

We say that Σ is *satisfiable* iff there is some truth valuation t such that $\Sigma^t = 1$. When $\Sigma^t = 1$, t is said to *satisfy* Σ.

Tautology (validity), Contradiction

A formula A is a *tautology* iff for any truth valuation t, $A^t = 1$.

A formula A is a *contradiction* iff for any truth valuation t, $A^t = 0$.

Example. Let $A = (p \wedge q \Rightarrow r) \wedge (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$. Is A a tautology?

“Expressions”

$\neg 1$	0
$\neg 0$	1
$A \wedge 1$	A
$1 \wedge A$	A
$A \wedge 0$	0
$0 \wedge A$	0
$A \vee 1$	1
$A \vee 0$	A
$1 \vee A$	1
$0 \vee A$	A

Tautology (validity), Contradiction

A faster way to evaluate a propositional formula is by using valution *trees* and “expressions”.

Example. Show that $A = (p \wedge q \Rightarrow r) \wedge (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ is a A a tautology.

Tautological Consequence

Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$, and $\mathcal{A}$ are propositions. Deductive logic studies whether $\mathcal{A}$ is *deducible* from $\mathcal{A}_1, \ldots, \mathcal{A}_n$.

Suppose $\Sigma \subseteq Form(\mathcal{L}^p)$ and $A \in Form(\mathcal{L}^p)$. We say that A is a *tautological consequence* of Σ (that is, of the formulas in Σ), written as $\Sigma \models A$, iff for any truth valuation t, $\Sigma^t = 1$ implies $A^t = 1$.

Note that $\Sigma \models A$ is not a formula.

Tautological Consequence

We write $\Sigma \not\models A$ for "not $\Sigma \models A$". That is, there exists some truth valuation t such that $\Sigma^t = 1$ and $A^t = 0$.

$\emptyset \models A$ means that A is a tautology. (why?)

Example. $A \Rightarrow B$, $B \Rightarrow C \models A \Rightarrow C$.

Example. $(A \Rightarrow \neg B) \vee C$, $B \wedge \neg C$, $A \Leftrightarrow C \not\models A \wedge (B \Rightarrow C)$.

Associativity of Commutativity

$$A \wedge B \equiv B \wedge A$$

$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$$

$$A \vee B \equiv B \vee A$$

$$(A \vee B) \vee C \equiv A \vee (B \vee C)$$

Tautological Consequence

Theorem.

[1] $A_1, \ldots, A_n \models A$ iff $\emptyset \models A_1 \wedge \cdots \wedge A_n \Rightarrow A_n$

[2] $A_1, \ldots, A_n \models A$ iff $\emptyset \models A_1 \Rightarrow (\ldots(A_n \Rightarrow A)\ldots)$

Tautological Consequence

Lemma.

If $A \equiv A'$ and $B \equiv B'$, then

1. $\neg A \equiv A'$
2. $A \wedge B \equiv A' \wedge B'$
3. $A \vee B \equiv A' \vee B'$
4. $A \Rightarrow B \equiv A' \Rightarrow B'$
5. $A \Leftrightarrow B \equiv A' \Leftrightarrow B'$

Replaceability

Theorem. If $B \equiv C$ and A' results from A by replacing some (not nessessarily all) occurrences of B in A by C, then $A \equiv A'$.

Duality

Theorem. Suppose A is a formula composed of atoms and the connectives $\neg$, $\wedge$, and $\vee$ by the formation rules concerned, and A' results by exhchanging in A, $\wedge$ for $\vee$ and each atom for its negation. Then $A' \equiv \neg A$. (A' is the *dual* of A)

Adequate Sets

Formulas $A \Rightarrow B$ and $\neg A \vee B$ are tautologocally equivalent. Then $\Rightarrow$ is said to be *definable* in terms of (or *reducible*) $\neg$ and $\vee$.

Let f and g be two n-ary connectives.

We shall write $f A_1 \ldots A_n$ for the formula formed by an n-ary connective f connecting formulas $A_1, \ldots, A_n$.

Question. Given $n \geq 1$, how many n-ary connectives exist?

Adequate Sets

Example. Suppose f_1, f_2, and f_3 are distinct unary connectives. They have the following truth tables:

A	$f_1 A$	$f_2 A$	$f_3 A$	$f_4 A$
1	1	1	0	0
0	1	0	1	0

A set of connetives is said to be *adequate* iff any n-ary ($n \geq 1$) connective can be defined in terms of them.

Adequate Sets

Theorem. $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Corollary. $\{\wedge, \neg\}$, $\{\vee, \neg\}$, $\{\Rightarrow, \neg\}$ are adequate.