Logic and Computation

CS245

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Sequent Calculus

Agenda

- Sequents
- LK Inference Rules
- Examples
- Soundness and Completeness

Sequents (syntax)

A *sequent* is a pair (Γ, Δ) of finite sets of formulae.

We will write a sequent as $\Gamma \vdash \Delta$ and drop the set brackets around the sets of formulae. Γ and Δ are *sets* of formulae. A and B will represent a single formula.

(Sometimes $\vdash$ is written as $\hookrightarrow$ or just $\rightarrow$.)

In sequents, we write Γ, A for $\Gamma \cup \{A\}$ and Γ, Δ for $\Gamma \cup \Delta$.

Sequents

'$\vdash$' is called 'proves' or 'turnstile'.

In a sequent $\Gamma \vdash \Delta$, the formulas on the left are called the *antecedent*, and the formulas on right-hand side are called the *succedent*.

When there is nothing on the LHS or RHS of the arrow, we assume it is the empty set of formulae.

Meaning of Sequents

A sequent asserts: if all the formulae on the left of the arrow are true, then at least one of the formulae on the right are true.

Formally, let $\Gamma = \{A_1, \ldots, A_n\}$ and $\Delta = \{B_1, \ldots, B_k\}$.

$A_1, \ldots, A_n \ \vdash \ B_1, \ldots, B_k$ means:

$$\vdash \ (A_1 \wedge \cdots \wedge A_n) \ \Rightarrow \ (B_1 \vee \cdots \vee B_k)$$

Meaning of Sequents

$$\vdash \ (A_1 \wedge \cdots \wedge A_n) \ \Rightarrow \ (B_1 \vee \cdots \vee B_k)$$

means that if there exists i suth that $A_i = false$, then the sequent is true.

Likewise, if there exists j suth that $A_j = true$, then the sequent is true.

LK System Axioms

$X \vdash X$ (Identity)

$$\mathbf{false} \vdash$$

$$\vdash \mathbf{true}$$

Sequent Calculus Inference Rules

Structural Rule (Thinning)

If $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$ then:

$$\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_2 \vdash \Delta_2}$$

Thinning is like *precondition strengthening* and *postcondition weakening* for those familiar with that terminology.

Adding formulae on the RHS of the sequent is adding them to a disjunction, so this is weakening the RHS formulae.

Adding formulae on the LHS of the sequent is adding them to a conjunction, so this is strengthening the LHS formulae.

Sequent Calculus Rules

Negation Rules

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}(\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}(\neg R)$$

Conjunction Rules

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}$$

Disjunction Rules

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}$$

Sequent Calculus Rules

Implication Rules

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B}(\Rightarrow R) \qquad \frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Delta_1, \Delta_2}(\Rightarrow L)$$

The right-hand rule is similar to *modus ponens*. The idea is that if X can be derived, then from $X \Rightarrow Y$, Y can be derived, and therefore whatever can be derived from Y can be derived. If Γ_2, and Δ_1 are empty:

$$\frac{\Gamma_1 \vdash A \qquad B \vdash \Delta_2}{\Gamma_1, A \Rightarrow B \vdash \Delta_2}$$

Cut Rule

This a derived rule:

$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

Question: How is it derived?

Proofs in the Sequent Calculus

Definition. A proof is a *tree* labelled with sequents (generally written with the root at the bottom), such that: if node N is labelled with $\Gamma \vdash \Delta$, then if N is a leaf node, $\Gamma \vdash \Delta$ must be an axiom; if N has children, their labels must be the premises from which $\Gamma \vdash \Delta$ follows by one of the rules. The label on the root node is the sequent that is proved.

Definition. A formula X is a theorem of the sequent calculus if the sequent $\vdash X$ has a proof, i.e.,

$$\vdash X$$

The sequent calculus for propositional logic is both sound and complete.

Example

Show in sequent calculus

$\vdash A \vee \neg A$

1.	$A \vdash A$	(id)
2.	$\vdash \neg A, A$	$(\neg R)$
3.	$\vdash A \vee \neg A, A$	(Disjunction))
4.	$\vdash A \vee \neg A, \neg A \vee A$	(Disjunction))
5.	$\vdash A \vee \neg A$	

Example

Show $\vdash A \Rightarrow (B \Rightarrow A)$

1.	$A \vdash A$	(id)
2.	$A, B \vdash A$	(Thinning)
3.	$A \vdash B \Rightarrow A$	(Implication ($\Rightarrow R$))
4.	$\vdash A \Rightarrow (B \Rightarrow A)$	(Implication ($\Rightarrow R$))

Example

Show in sequent calculus

$$\vdash (A \Rightarrow (B \Rightarrow C)) \Leftrightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$