Logic and Computation

CS245

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First-order (Predicate) Logic

Agenda

- Syntax
- Semantics
- Proof System
- Soundness and Completeness

Motivation

In propositional logic, only the logical forms of compound propositions are analyzed.

We need some way to talk about *individuals* (also called *objects*.

Motivation

For example:

- For any natural number n, there is a prime number greater than n.
- 2^{100} is a natural number.
- There is a prime number greater than 2^{100}

First-order logic (also called *predicate logic*) gives us means to express and reason about objects.

Motivation

We also need ability to define *sets* by set comprehension $\{x \mid I \models \varphi(x)\}$

And incorporate *relations*.

As well as *properties* of interpretations (e.g., all graphs that are ...)

Structure of FOL

First-order logic is a scientific theory with the following ingredients:

- Domain of objects (individuals) (e.g., the set of natural numbers)
- Designated individuals (e.g., ‘0’)
- Functions (e.g., ‘+’ and ‘.’)
- Relations (e.g., ‘=’)

Structure of FOL

We use variables that range over the domain to make general statements:

For all $x, x^2 \geq 0$.

and in expressing conditions which individuals may or may not satisfy:

$$x + x = x.x$$

This condition is satisfied only by 0 and 2.

Structure of FOL

One can use connectives to form compound propositions.

We use the terms "*for all* " and "*there exists*" frequently (called *quantifiers*). For example:

- For all $\epsilon > 0$, there exists some $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - b| < \epsilon$.

"For all" is called the *universal quantifier* and "there exists" is called the *existential quantifier*.

Propositions and Functions

$$4 \text{ is even}$$

is a proposition since 4 is an individual in $\mathbb{N}$. If we replace 4 by a variable x ranging over $\mathbb{N}$, then

$$x \text{ is even}$$

is not a proposition and has no truth value. It is a proposition function.

A *proposition function* on a domain D is an n-ary function mapping D^n into $\{0, 1\}$.

Prefixing Quantifiers

Consider:

For all x, x is even.
There exists x, such that x is even.

Since x ranges over $\mathbb{N}$, they mean:

For all natural numbers x, x is even.
There exists a natural number x, such that x is even.

These have truth values!

Bound and Quantified Variables

Variables occurring in proposition functions are *free variables*.

Quantified variables are called *bound variables*.

Quantifiers for Finite Domains

The universal and existential quantifiers may be interpreted respectively as generalization of conjunction and disjunction. If the domain $D = \{\alpha_1, \ldots, \alpha_k\}$ is finite then:

For all x st. $f(x)$ iff $R(\alpha_1)$ and ... and $R(\alpha_k)$

There exists x st. $R(x)$ iff $R(\alpha_1)$ or ... or $R(\alpha_k)$

where R is a property.

FOL Language $\mathcal{L}$

1. Constant (individual) symbols (CS):
   $c, d, c_1, c_2, \ldots, d_1, d_2 \ldots$
2. Function Symbols (FS):
   $f, g, h, f_1, f_2, \ldots, g_1, g_2$
3. Variables (VS): $x, y, z, x_1, x_2, \ldots, y_1, y_2 \ldots$
4. Predicate (Relational) Symbols (PS):
   $P, Q, P_1, P_2, \ldots, Q_1, Q_2, \ldots$
5. Logical Connectives: $\neg, \wedge, \vee, \Rightarrow$
6. Quantifiers: $\forall$ (for all) and $\exists$ (there exists)
7. Punctuation: ‘(’, ‘)’, ‘.’, and ‘,’.

Example

- 0: constant '0'
- S: function (successor) $S(x)$ stands for: '$x + 1$'
- Eq: relation (equality) $Eq(x, y)$ stands for: '$x = y$'
- plus: function (addition) $\mathrm{plus}(x, y)$ stands for: '$x + y$'

$$\forall x.\mathrm{Eq}(\mathrm{plus}(x, S(S(0))), S(S(x)))$$

means "Adding two to a number results in the second successor of that number"

Example

$$\forall x.\forall y.\mathrm{Eq}(\mathrm{plus}(x, y), \mathrm{plus}(y, x))$$

means “Addition is commutative.”

$$\neg\exists x.\mathrm{Eq}(0, S(x))$$

means “0 is not the successor of any number.”

Terms

The set $Term(\mathcal{L})$ of *terms* of $\mathcal{L}$ is defined using the following rules:

- All constants in CS are terms
- All variables in VS are terms
- it $t_1, \ldots, t_n \in Term(\mathcal{L})$ and f is an n-ary function, then $f(t_1, \ldots, t_n) \in Term(\mathcal{L})$.

For example, 0, x, and y are terms and so are $S(0)$, $\text{plus}(x, y)$.

Syntax of FOL - Atoms

An expression of $\mathcal{L}$ is an *atom* in $Atoms(\mathcal{L})$ iff it is of one of the forms $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms in $Term(\mathcal{L})$.

Syntax of FOL - Formulas

We define the set $Form(\mathcal{L})$ of first-order logic formulas inductively as follows:

1. $Atom(\mathcal{L}) \subseteq Form(\mathcal{L})$
2. If $A \in Form(\mathcal{L})$, then $\neg A \in Form(\mathcal{L})$
3. If $A, B \in Form(\mathcal{L})$, then $(A * B) \in Form(\mathcal{L})$, where $* \in \{\wedge, \vee, \Rightarrow\}$
4. If $A \in Form(\mathcal{L})$ and $x \in VS$, then $(\forall x.A) \in Form(\mathcal{L})$ and $(\exists x.A) \in Form(\mathcal{L})$

Example

How is the following formula generated?

$$\forall x.(F(b) \Rightarrow \exists y.(\forall z.G(y,z) \ \vee \ H(u,x,y)))$$

Free and Bound Variables

Let $A \in Form(\mathcal{L})$. We define the set FV(A) of *free variables* of A as follows:

1. $\{x \mid x$ appears in t_i for some $0 < i \leq ar(P)\}$, for $A = P(t_1, \ldots, t_{ar(P)})$
2. FV(A) for $B = (\neg A)$
3. FV$(A) \cup$ FV(B) for $C = (A * B)$, where $* \in \{\wedge, \vee, \Rightarrow\}$
4. FV$(A) - \{x\}$ for $B = (\forall x.A)$ or $B = (\exists x.A)$

Variables not in FV(A) are called *bound variables*.

Closed Formulas

A first-order formula $A \in Form(\mathcal{L})$ is *closed* (also called a *sentence*) if FV$(A) = \{\}$.

Scope

If $\forall x.A(x)$ or $\exists x.A(x)$ is a segment of B, $A(x)$ is called the *scope* in B of the $\forall x$ or $\exists x$ on the left of $A(x)$.

In the following formula:

$$\exists x.\forall y.\exists z.F(x, y, z)$$

what is the scope of $\forall y$?

FOL Interpretations

First-order formulas are intended to express propositions (i.e, true/false valuation). This is accomplished by *interpretations*

Interpretations for the propositional language are simple: they consist of assigning values to the proposition symbols.

The first-order language includes more ingrediants and, hence, the interpretations for it are more complicated.

FOL Interpretations

A first order *interpretation* I is a tuple $(D, (.)^I)$:

- D is a non-empty set called the *domain* (or *universe*); and
- $(.)^I$ is an *interpretation function* that maps
  - constant symbols $c \in CS$ to individuals $c^I \in D$;
  - function symbols $f \in FS$ to functions $f^I : D^{ar(f)} \rightarrow D$; and
  - predicate symbols $P \in PS$ to relations $P^I \subseteq D^{ar(P)}$.

FOL Interpretations

Let

$$f(g(a), f(b, c))$$

be a term. Let individuals a, b, and c be interpreted as 4, 5, and 6 in $\mathcal{N}$ and functions f and g are respectively as addition and squaring. Then, the above term is interpreted as

$$4^2 + (5 + 6)$$

which is the individual 27 in $\mathcal{N}$.

FOL Interpretations

Let functions f and g are respectively addition and squaring and P be the equality relation. Let

$$P(f(g(a), g(b)), g(c))$$

be a closed formula, where individuals a, b, and c be interpreted as 4, 5, and 6 in $\mathcal{N}$.Then, the above predicate is interpreted as the false proposition (why?).

FOL Valuations

Now, consider the non-closed formula:

$$P(f(g(u), g(b)), g(w))$$

where only b is interpreted as 5. One can interpret this formula by:

$$x^2 + 5^2 = y^2$$

where x and y are free variables. This is not a proposition, but a binary proposition function in $\mathbb{N}$. One can obtain a truth value by assigning individuals in $\mathbb{N}$ to x and y. This is called a *valuation*.

FOL Valuations

A *valuation* θ (also called an *assignment*) is a mapping from VS, the set of variables, to domain D.

For example, for the non-closed formula

$$x^2 + 4^2 = y^2$$

$\theta(x) = 3$ and $\theta(y) = 5$ evaluates the formula to the true proposition.

Meaning of Terms

Let I be a first order interpretation and θ a valuation. For a term t in $Term(\mathcal{L})$, we define interpretation of t, t^I , as follows:

1. $c^{I,\theta} = c^I$ for $t \in CS$ (i.e., t is a constant);
2. $x^{I,\theta} = \theta(x)$ for $t \in VS$ (i.e., t is a variable); and
3. $f(t_1, \ldots, t_{ar(f)})^{I,\theta} = f^I((t_1)^{I,\theta}, \ldots, (t_{ar(f)})^{I,\theta})$, otherwise (i.e., for t a functional term).

Satisfaction Relation

The *satisfaction relation* $\models$ between an interpretation I, a valuation θ, and a first-order formula φ is defined as:

- $I, \theta \models P(t_1, \ldots, t_{ar(P)})$ iff $\langle (t_1)^{I,\theta}, \ldots, (t_{ar(P)})^{I,\theta} \rangle \in P^I$ for $P \in PS$
- $I, \theta \models \neg\varphi$ if and only if $I, \theta \models \varphi$ is not true
- $I, \theta \models \varphi \wedge \psi$ if and only if $I, \theta \models \varphi$ and $I, \theta \models \psi$
- $I, \theta \models (\forall x.\varphi)$ if and only if $I, \theta([x = v]) \models \varphi$ for all $v \in D$

where the valuation $[x = v](y)$ is defined to be v when $x = y$ and θ otherwise.

Satisfaction Relation

One can also trivially define the following:

- $I, \theta \models \varphi \vee \psi$ if and only if $I, \theta \models \varphi$ or $I, \theta \models \psi$
- $I, \theta \models (\exists x.\varphi)$ if and only if $I, \theta([x = v]) \models \varphi$ for some $v \in D$

where the valuation $[x = v](y)$ is defined to be v when $x = y$ and θ otherwise.

Some Remarks

1. $\langle (t_1)^{I,\theta}, \ldots, (t_{ar(P)})^{I,\theta} \rangle \in P^I$ means that $(t_1)^{I,\theta}, \ldots, (t_{ar(P)})^{I,\theta}$ is in relation P^I
2. If $A(x)$ is a variable with no free occurrence of u an $A(u)$ is a formula with no free occurrence of x, then $A(x)$ and $A(u)$ have the same intuitive meaning.
3. For the same reason, $\forall x.A(x)$ and $\forall u.A(u)$ have the same meaning.

Relevance Lemma

Let φ be a first-order formula, I be an interpretation, and θ_1 and θ_2 be two valuations such that $\theta_1(x) = \theta_2(x)$ for all $x \in VS$. Then,

$$I, \theta_1 \models \varphi \text{ iff } I, \theta_2 \models \varphi$$

Proof by structural induction.

Satisfiability and Validity

$\Sigma \subseteq Form(\mathcal{L})$ is *satisfiable* iff there is some interpretation I and valuation θ, such that $I, \theta \models \varphi$ for all $\varphi \in \Sigma$.

A formula $\varphi \in Form(\mathcal{L})$ is *valid* iff for all interpretations I and valuations θ, we have $I, \theta \models \varphi$

Example

Let $\varphi = P(f(g(x), g(y)), g(z))$ be a formula. The formula is satisfiable:

- $f^I =$ summation
- $g^I =$ squaring
- $P^I =$ equality
- $\theta(x) = 3, \theta(y) = 4, \theta(z) = 5$

φ is not valid. (why?)

Logical Consequence

Suppose $\Sigma \subseteq Form(\mathcal{L})$ and $\varphi \in Form(\mathcal{L})$. We say that φ is a *logical consequence* of Σ (that is, of the formulas in Σ), written as $\Sigma \models \varphi$, iff for any interpretation I and valuation θ, we have $I, \theta \models \Sigma$ implies $I, \theta \models \varphi$.

$\models \varphi$ means that φ is valid.

Example

Show that $\models \forall x.(\varphi \Rightarrow \psi) \Rightarrow ((\forall x.\varphi) \Rightarrow (\forall x.\psi))$

Proof by contradiction: there exists I and θ st.

$I, \theta \not\models \forall x.(\varphi \Rightarrow \psi) \Rightarrow ((\forall x.\varphi) \Rightarrow (\forall x.\psi))$
$I, \theta \models \forall x.(\varphi \Rightarrow \psi)$
$I, \theta \models \forall x.\varphi$
$I, \theta \not\models \forall x.\psi$

$I, \theta([x = v]) \models \varphi$
$I, \theta([x = v]) \not\models \psi$
$I, \theta([x = v]) \not\models \varphi \Rightarrow \psi$
$I, \theta \not\models \forall x.(\varphi \Rightarrow \psi)$ (contradiction)

Example

Show that $\forall x.\neg A(x) \models \neg\exists x.A(x)$

Proof by contradiction: there exists I and θ st.
$I, \theta \models \forall x.\neg A(x)$ and $I, \theta \not\models \neg\exists x.A(x)$
$I, \theta \models \exists x.A(x)$

$I, \theta([x = v]) \models \neg A(x)$ for all v
$I, \theta([x = v]) \models A(x)$ for some v

Contradiction!

Example

Show that $((\forall x.\varphi) \Rightarrow (\forall x.\psi)) \not\models \forall x.(\varphi \Rightarrow \psi)$

Replacability and Duality

Theorem. If $B \equiv C$ and A' results from A by *replacing* some (not necessarily all) occurrences of B in A by C, then $A \equiv A'$.

Theorem. Suppose A is a formula composed of atoms and the connectives $\neg$, $\wedge$, and $\vee$ by the formation rules concerned, and A' results by exchanging in A, $\wedge$ for $\vee$ and each atom for its negation. Then $A' \equiv \neg A$. (A' is the *dual* of A)

Substitution

1. For a term t_1, $(t_1)^x_t$ is t_1 with each occurrence of the variable x replaced by the term t.
2. For $\varphi = P(t_1, \ldots, t_{ar(P)})$, $(\varphi)^x_t = P\big((t_1)^x_t, \ldots, (t_{ar(P)})^x_t\big)$.
3. For $\varphi = (\neg\psi)$, $(\varphi)^x_t = (\neg(\psi)^x_t)$;
4. For $\varphi = (\psi \to \eta)$, $(\varphi)^x_t = ((\psi)^x_t \to (\eta)^x_t)$, and
5. for $\varphi = (\forall y.\psi)$, there are two cases:
   - if x is y, then $(\varphi)^x_t = \varphi = (\forall y.\psi)$, and
   - otherwise, then $(\varphi)^x_t = (\forall z.(\psi^y_z)^x_t)$, where z is any variable that is not free in t or in φ.

Substitution

In the last case above, the additional substitution $(.)^y_z$ (i.e., renaming the variable y to z in ψ) is needed in order to avoid an accidental *capture of a variable* by the quantifier (i.e., capture of any y that is possibly free in t).

Substitution Lemma

$$\models \forall x.\varphi \Rightarrow \varphi^x_t$$

$$I,\theta \models \varphi^x_t \textsf{ iff } I,\theta[x = (t)^{I,\theta}] \models \varphi$$

FOL Hilbert System

Ax1 $\langle \forall^*(\varphi \to (\psi \to \varphi)) \rangle$;

Ax2 $\langle \forall^*((\varphi \to (\psi \to \eta)) \to ((\varphi \to \psi) \to (\varphi \to \eta))) \rangle$;

Ax3 $\langle \forall^*(((\neg\varphi) \to (\neg\psi)) \to (\psi \to \varphi)) \rangle$;

Ax4 $\langle \forall^*(\forall x.(\varphi \to \psi)) \to ((\forall x.\varphi) \to (\forall x.\psi)) \rangle$;

Ax5 $\langle \forall^*(\forall x.\varphi) \to \varphi^x_t \rangle$ for $t \in$ T a term;

Ax6 $\langle \forall^*(\varphi \to \forall x.\varphi) \rangle$ for $x \notin$ FV(φ); and

MP $\langle \varphi, (\varphi \to \psi), \psi \rangle$.

where $\forall^*$ is a finite sequence of universal quantifiers (e.g., $\forall x_1.\forall y.\forall x$).

Generalization of Axioms (why $\forall *$)

Show that if $\Phi \vdash \varphi$ and $x \notin$ FV(Φ), then $\Phi \vdash \forall x.\varphi$.

Proof by structural induction.

Base case: φ is an axiom. Then, $\Phi \vdash \forall x.\varphi$.

Induction step (1): $\varphi \in \Phi$

$\Phi \vdash \varphi$
$\vdash \varphi \Rightarrow \forall x.\varphi$
$\Phi \vdash \forall x.\varphi$ (MP and $x \notin$ FV(φ))

Generalization of Axioms (why $\forall *$)

Induction step (2): $\psi \Rightarrow \varphi$

1. $\Phi \vdash \psi$, $\Phi \vdash \psi \Rightarrow \varphi$ (Induction hyp.)
2. $\Phi \vdash (\forall x.\psi)$ (Ax_6)
3. $\Phi \vdash \forall x.(\psi \Rightarrow \varphi)$ (Ax_6)
4. $\Phi \vdash (\forall x.\psi) \Rightarrow \forall x.\varphi)$ (Ax_5)
5. $\Phi \vdash (\forall x.\varphi)$ (MP)

Example 1

Show that $\vdash \forall x.\forall y.\varphi \Rightarrow \forall y.\forall x.\varphi$

1. $\forall x.\forall y.\varphi$	(Deduction theorem)
2. $\forall x.\forall y.\varphi \Rightarrow (\forall y.\varphi)^x_t$	(Ax_5)
3. $(\forall y.\varphi)^x_t$	(MP)
4. $(\forall y.\varphi)^x_t \Rightarrow ((\varphi)^x_t)^y_{t'}$	(Ax_5)
5. $((\varphi)^x_t)^y_{t'}$	(MP)
6. $((\varphi)^x_t)^y_{t'} \Rightarrow \forall x.(\varphi)^x_t$	(Ax_6)
7. $\forall x.(\varphi)^x_t$	(MP)
8. $\forall x.(\varphi)^x_t \Rightarrow \forall y.\forall x.\varphi$	(Ax_6)
9. $\forall y.\forall x.\varphi$	(MP)

Example 2

Show that $\vdash A(a) \Rightarrow \exists x.A(x)$

1. $\forall x.\neg A(x) \Rightarrow \neg A(a)$ (Ax_5)
2. $A(a) \Rightarrow (\neg\forall x.\neg A(x))$ (Ax_3)
3. $A(a) \Rightarrow \exists x.A(x)$ (Definition of $\exists$)

Example 3

Show that

$\vdash \forall x.(A(x) \Rightarrow B(x)) \Rightarrow (\forall x.A(x) \Rightarrow \forall x.B(x))$

1. $\forall x.(A(x) \Rightarrow B(x))$ (Assumption)
2. $\forall x.A(x)$ (Assumption)
3. $\forall x.A(x) \Rightarrow A(a)$ (Ax_5)
4. $A(a)$ (MP 2, 3)
5. $\forall x.(A(x) \Rightarrow B(x)) \Rightarrow (A(a) \Rightarrow B(a))$ (Ax_5)
6. $A(a) \Rightarrow B(a)$ (MP 1, 5)
7. $B(a)$ (MP 4, 6)
8. $B(a) \Rightarrow \forall x.B(x)$ (Ax_6)

Example 4

Show that $\exists x.\forall y.A(x,y) \Rightarrow \forall y.\exists x.A(x,y)$.

Soundness of FOL Hilbert System

Step 1: *Satisfiability and validity in domain*

Suppose $\Sigma \subseteq Forma(\mathcal{L})$, $A \in Form(\mathcal{L})$, and D is a domain.

1. Σ is satisfiable in D iff there is some model I, θ over D such that $I, \theta \models \varphi$ for all $\varphi \in \Sigma$.
2. A is valid in D iff for all models I, θ over D, we have $I, \theta \models A$.

Soundness of FOL Hilbert System

Theorem. Suppose formula A contains no equality symbol and $|D| \leq |D_1|$.

- If A is satifiable in D, then A is satisfiable in D_1.
- If A is valid in D_1, then A is valid in D.

Soundness of FOL Hilbert System

Theorem (Soundness).

- If $\Sigma \vdash A$, then $\Sigma \models A$.
- If $\vdash A$, then $\models A$.
  (That is, every formally provable formula is valid.)

Consistency

$\Sigma \subseteq Form(\mathcal{L})$ is *consistent* iff there is no $A \in Form(\mathcal{L})$ such that $\Sigma \vdash A$ and $\Sigma \vdash \neg A$.

Consistency is a syntactical notion

Theorem. If Σ is satifiable, then Σ is consistent.

Maximal Consistency

$\Sigma \subseteq Form(\mathcal{L})$ is *maximal consistent* iff

1. Σ is consistent
2. for any $A \in Form(\mathcal{L})$ such that $A \notin \Sigma$, $\Sigma \cup \{A\}$ is inconsistent.

Lemma. Suppose Σ is maximal consistent. Then, $A \in \Sigma$ iff $\Sigma \vdash A$.

Lindenbaum Lemma. Any consistent set of formulas can be extended to some maximal consisten set.

Completeness of FOL

Theorem. Suppose $\Sigma \subseteq Form(\mathcal{L})$. If Σ is consistent, then Σ is satisfiable.

Theorem. Suppose $\Sigma \subseteq Form(\mathcal{L})$. and $A \in Form(\mathcal{L})$. Then

1. If $\Sigma \models A$, then $\Sigma \vdash A$.
2. If $\models A$, then $\vdash A$.

FOL with Equality

Let $\approx$ be a binary predicate symbol (written in infix). We define the *First-Order Axioms of Equality* as follows:

EqId	$\langle \forall x.(x \approx x) \rangle$;
EqCong	$\langle \forall x.\forall y.(x \approx y) \rightarrow (\varphi\,{}^{z}_{x} \rightarrow \varphi\,{}^{z}_{y}) \rangle$;

FOL with Equality

Gödel's Completeness Theorem. Hilbert system *with (axiomatized) equality* is

- sound; i.e., if $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$ and
- complete; i.e, if $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$

with respect to first-order logic *with (true) equality*.

Definability

Let $I = (D, (.)^I)$ be a first-order interpretation and φ a first-order formula. A set S of k-tuples over D, $S \subseteq D^k$, is *defined* by the formula φ if

$$S = \{(\theta(x_1), \ldots, \theta(x_k)) \mid I, \theta \models \varphi\}$$

A set S is *definable* in first-order logic if it is defined by some first-order formula φ.

Definability

Let Σ be a set of first-order sentences and $\mathcal{K}$ a set of interpretations. We say that Σ defines $\mathcal{K}$ if

$$I \in \mathcal{K} \text{ if and only if } I \models \Sigma.$$

A set $\mathcal{K}$ is *(strongly) definable* if it is defined by a (finite) set of first-order formulas Σ.

Compactness in FOL

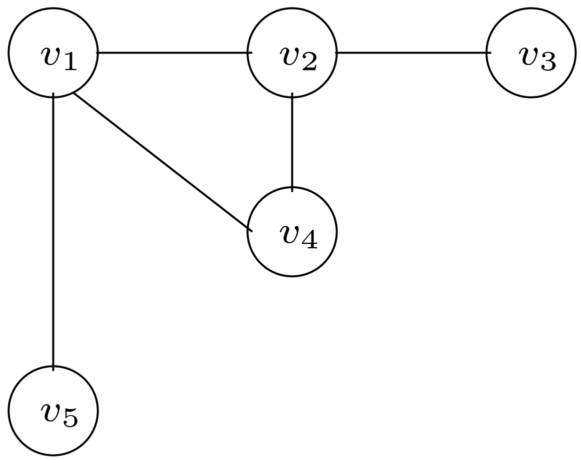
Theorem. $\Sigma \subseteq Form(\mathcal{L})$ is satisfiable iff every finite subset of Σ is satisfiable.

Corollary. $\Sigma \subseteq Form(\mathcal{L})$ is satisfiable in a finite domain, then Σ is satisfiable in an infinite domain.

Corollary. The class of interpretations with finite domain is not definable in first-order logic.

Graphs

An undirected *graph* is a tuple (V, E), where V is a set of vertices and E is a set of edges. An edge is a pair (v_1, v_2), where $v_1, v_2 \in V$.



$$V = \{v_1, v_2, v_3, v_4, v_5\}$$
$$E = \{(v_1, v_2), (v_2, v_3), (v_2, v_4), (v_1, v_4), (v_1, v_5)\}$$

Graphs in FOL

If $(v_1, v_2) \in E$, we say that v_1 is *adjacent* to v_2.

Adjacency in a graph can be expressed by a binary relation. Thus, relation $E(v_1, v_2)$ is interpreted as "v_1 is adjacent to v_2". A graph is any model of the following 2 axioms:

1. $\forall x.\forall y.E(x, y) \Rightarrow E(y, x)$ ("if x is adjacent to y, then y is adjacent to x")
2. $\forall x.\neg E(x, x)$ ("no x is adjacent to itself")

Graphs in FOL

We can express many properties of a graph in the language of first-order logic.

For instance, the property "G contains a triangle" is the following formula:

$$\exists x.\exists y.\exists z.(E(x,y) \wedge E(y,z) \wedge E(z,x))$$

Example

Define first-order formulas for :

- A graph has *girth* of size 4
- A graph is 3-colorable

Graph Connectivity in FOL

We cannot express graph *connectivity* in FOL (i.e., graph connectivity is not definable in FOL).

Proof.

- Let predicate C express "G is a connected graph". We add constants s and t vertices.
- For any k, let L_k be the proposition "there is no path of length k between s and t". For example,

$$L_3 = \neg\exists x.\exists y.(E(s,x) \wedge E(x,y) \wedge E(y,t))$$

Graph Connectivity in FOL

- Now consider the set of propositions

$$\Sigma = \{\text{axiom}(1), \text{axiom}(2), C, L_1, L_2, \ldots\}$$

- Σ is finitely satisfiable: there do exist connected graphs with s and t, that are connected by an arbitrarily long path. This is because any finite subset $F \subset \Sigma$ must have bounded k's, such a graph satisifes F.

Graph Connectivity in FOL

- By the compactness theorem, Σ is satisfiable; i.e., there exists some model G of all propositions Σ, which is a graph that cannot be connected by a path of length k, for any k, for all k.
- This is clearly wrong. In a connected graph, any 2 nodes are connected by a path of finite length!

Cyclic Graphs in FOL

Prove that there is no first order sentence φ with the property that for each undirected graph G, there is $G \models \varphi$ iff every vertex of G belongs to a (finite) cycle in the graph.

Cyclic Graphs in FOL

- Assume that such a set Σ of sentences exists. Extend the signature with a new constant c and extend Σ with the set

$$\{\neg\exists v_1.\exists v_2 \ldots \exists v_{n-1}.\exists v_n.E(c, v_1) \wedge E(v_1, v_2) \wedge E(v_2, v_3) \wedge \cdots \wedge E(v_{n-1}, v_n) \wedge E(v_n, c) | n \in \mathbb{N}\}.$$

Cyclic Graphs in FOL

- The extended set satisfies the conditions of the compactnes theorem: its every finite subset is satisfiable, since a finite number of added sentences prevent c from being on a cycle of a few finite sizes, so as a model one may take a finite cycle of sufficiently many vertices.
- As a result, the entire set has a model: a contradiction, as c does not belong to any cycle in it.

Löwenheim-Skolem's Theorems

Theorem 1. Suppose $\Sigma \subseteq Form(\mathcal{L})$.

1. Σ not containing equality is satisfiable iff Σ is satisfiable in a countably infinite domain.
2. Σ containing equality is satisfiable iff Σ is satisfiable in a countably infinite domain or in some finite domain.

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Theorem 2. Suppose $A \in Form(\mathcal{L})$.

1. A not containing equality is valid iff A is valid in a countably infinite domain.
2. A containing equality is valid iff A is valid in a countably infinite domain or in every finite domain.