Logic and Computation

CS245

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Modal Logic

Agenda

- Syntax
- Semantics
- Formal Deduction
- Soundness and Completeness

Modal Logic

Modal logic is a logic of modal notions.

Let A be a propostition. Can we express "A is necessary" and "A is possible" in propositional logic?

Necessity and *possibility* are basic modal notions.

Necessarily true propositions are said to be *necessary* and necessarily false propositions are said to be *impossible*.

Syntax

The *modal propositional logic language* $\mathcal{L}^{pm}$ is obained recursively as follows:

[1] $Atom(\mathcal{L}^{pm}) \subseteq Form(\mathcal{L}^{pm})$.

[2] If $A \in Form(\mathcal{L}^{pm})$, then

$$(\neg A), (\Box A) \in Form(\mathcal{L}^{pm})$$

[3] If $A, B \in Form(\mathcal{L}^{pm})$, then $(A * B) \in Form(\mathcal{L}^{pm})$, $*$ being any of $\wedge, \vee, \Rightarrow, \Leftrightarrow$.

Just for completeness

Formally, semantics is a function that mapps a formula to a value in $\{0, 1\}$ (also known as *truth table*).

$$\varphi_1 \vee \varphi_2 = \neg\varphi_1 \Rightarrow \varphi_2$$

$$\varphi_1 \wedge \varphi_2 = \neg(\varphi_1 \Rightarrow \neg\varphi_2)$$

$$\varphi_1 \Leftrightarrow \varphi_2 = (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_2 \Rightarrow \varphi_1)$$

$$\Diamond\varphi = \neg\Box\neg\varphi$$

Semantics

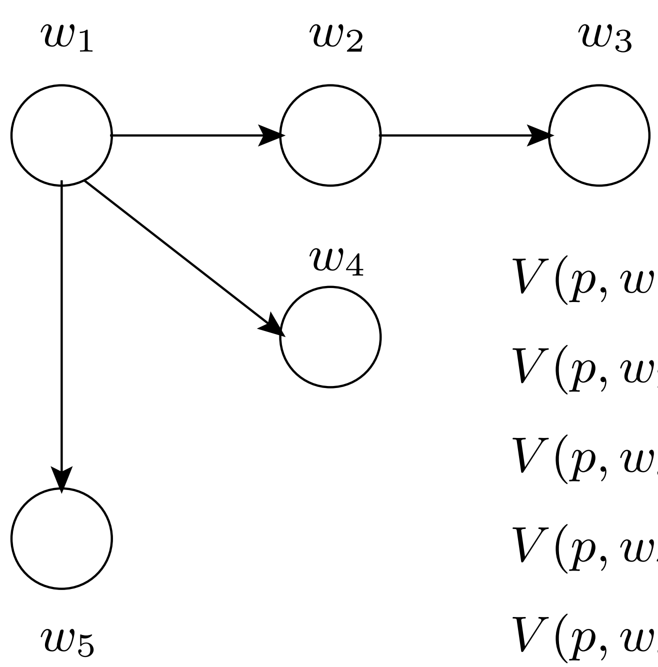
Kripke structures (possible worlds structures) are models of basic modal logic.

A Kripke structure (or *interpretation* is a triple $M = (W, R, V)$, where

- W is a non-empty set (possible *Worlds*)
- $R \subseteq W \times W$ is an *accessibility relation*
- $V : (Atom(\mathcal{L}^{pm}) \times W) \Rightarrow \{true, false\}$ is a *valuation function*.

Example

This is just a graph (W, R) with a function V which tells which propositional variables are true at which vertices.



$V(p, w_1) = \mathit{true}, V(q, w_1) = \mathit{false}$

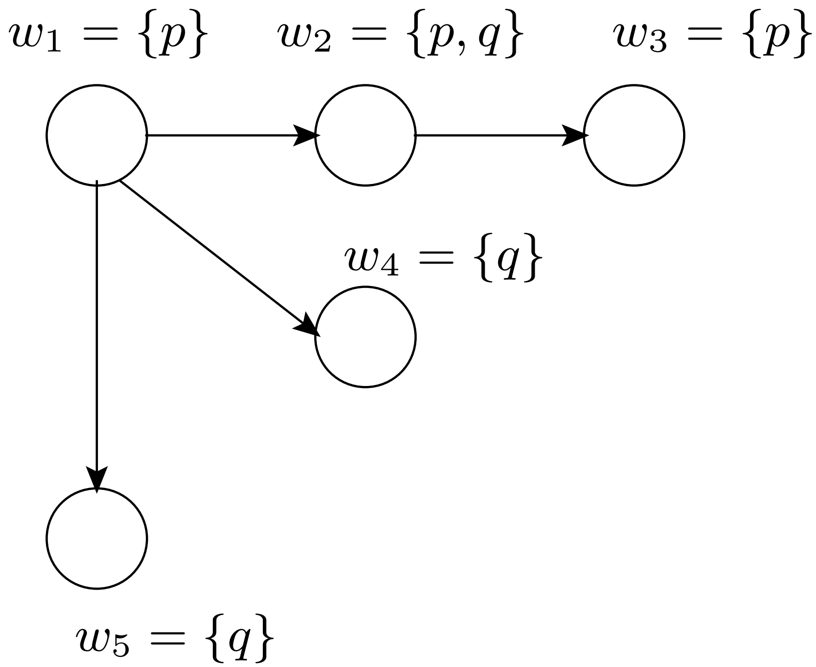
$V(p, w_2) = \mathit{true}, V(q, w_2) = \mathit{true}$

$V(p, w_3) = \mathit{true}, V(q, w_3) = \mathit{false}$

$V(p, w_4) = \mathit{false}, V(q, w_4) = \mathit{true}$

$V(p, w_5) = \mathit{false}, V(q, w_5) = \mathit{true}$

Example



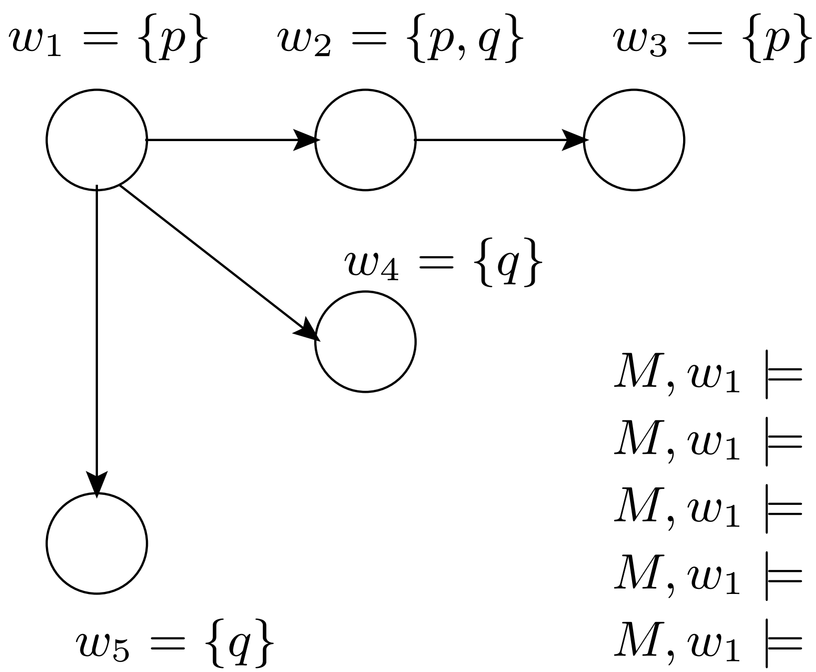
Semantics

Given $M = (W, R, V)$ and $w \in W$, we define what does it mean for a formula to be true (satisfied) in a world w of a model M:

$M, w \models p$	iff	$V(p, w) = true$
$M, w \models \neg\varphi$	iff	$M, w \not\models \varphi$
$M, w \models (\varphi \wedge \psi)$	iff	$(M, w \models \varphi) \ \wedge \ (M, w \models \psi)$
$M, w \models \Box\varphi$	iff	for all v accessible from w (for all v such that $R(w, v)$), $M, v \models \varphi$

The pair (W, R) is called the *frame* of M.

Example



$$M, w_1 \models \Box q$$
$$M, w_1 \models \neg\Box p$$
$$M, w_1 \models \neg\Box\neg p$$
$$M, w_1 \models \Diamond p$$
$$M, w_1 \models \Diamond\Box p$$

Pointed Models

A pair (M, w), such that $M, w \models \varphi$, is called a *(pointed) model* of φ. We define $\bmod(\varphi)$ to be

$$\bmod(\varphi) = \{(M, w) \mid (M, w) \models \varphi\}$$

In many presentations the term *model* and *interpretation* are used as synonyms; such a terminology, however, makes defining validity, satisfiability, and logical implication cumbersome.

Satifiability and Validity

A formula φ is *true* in a model M if it is satisfied in all of M's worlds

A formula φ is *valid* if it is true in all models.
I.e., If $M, w \models \varphi$ for all interpretations M and all $w \in W$

A formula is *satisfiable* if its negation is not valid (if it is satisfied in at least one world of one model).
I.e., if $M, w \models \varphi$ for some interpretation M and $w \in W$.

Equivalence and Logical Implication

Definitions of *logical implication* ($\Sigma \models \varphi$) and *equivalence*, and their properties are now the same as for propositional logic.

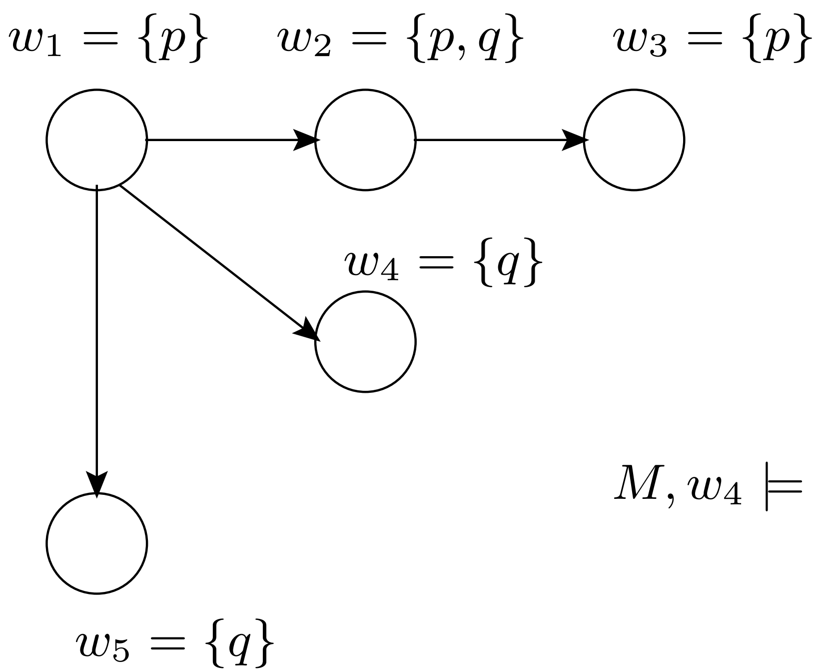
Example

$\Box p \Rightarrow \Box p$ is valid (just an example of a propositional tautology)

$\Box(p \Rightarrow p)$ is valid (because $p \Rightarrow p$ is true in all accessible worlds, wherever you are).

$\Box p \Rightarrow p$ is not valid (the set $\{\Box p, \neg p\}$ is satisfiable in some worlds).

Example



$$M, w_4 \models \Box p \wedge \neg p$$

Classes of Modal Logic

A modal formula *characterizes a class of frames* $\mathcal{F}$ if

- $M, w \models \varphi$ for all $M = (W, R, V)$ and $w \in W$, where the frame $(W, R) \in \mathcal{F}$, and
- $N, w \not\models \varphi$ for some $N = (W, R, V)$ and $w \in W$, where $(W, R) \notin \mathcal{F}$

Classes of Modal Logic

To make $\varphi_1 = \Box p \Rightarrow p$ valid, need to require that R is *reflexive*.

Then if $M, w \not\models p$, from $R(w, w)$ also $M, w \not\models \Box p$.

φ_1 characterizes reflexive relations (modal logic class T)

Classes of Modal Logic

- (Class S_4) $\Box p \Rightarrow \Box\Box p$ corresponds to *transitivity* of R (easier to see in $\Diamond$ form, $\Diamond\Diamond p \Rightarrow \Diamond p$: if you can get somewhere in two steps, you can get there is one step).
- (Class B) $p \Rightarrow \Box\Diamond p$ corresponds to *symmetry*
- (Class D) $\Box p \Rightarrow \Diamond p$ corresponds to *seriality* of R (for every world there is an accessible world)
- $\Diamond p \Rightarrow \Box\Diamond p$ corresponds to R being *euclidean* (*unique*)

Classes of Modal Logic

Show that in T:

$$\models \Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$$

Modal Logic and Proofs

Hilbert system extension (K):

Axioms:

- $Ax_1 - Ax_3$ of propositional Hilbert System
- MP
- $Ax_4 : \Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$
- NEC : If p is a theorem, then $\Box p$ is likewise a theorem.

Example

Prove that: $\Box(p \wedge q) \Rightarrow (\Box p \wedge \Box q)$

1.	$(p \wedge q) \Rightarrow p$	(Elimination)
2.	$(p \wedge q) \Rightarrow q$	(Elimination)
3.	$\Box((p \wedge q) \Rightarrow p)$	(NEC)
4.	$\Box((p \wedge q) \Rightarrow q)$	(NEC)
5.	$\Box((p \wedge q) \Rightarrow p) \Rightarrow (\Box(p \wedge q) \Rightarrow \Box p)$	(Ax_4)
6.	$(\Box(p \wedge q) \Rightarrow \Box p)$	(MP, 3, 5)
7.	$\Box p$	(Deduction Thrm, 6)
8.	$\Box((p \wedge q) \Rightarrow q) \Rightarrow (\Box(p \wedge q) \Rightarrow \Box q)$	(Ax_4)
9.	$(\Box(p \wedge q) \Rightarrow \Box q)$	(MP, 4, 8)
10.	$\Box q$	(Deduction Thrm, 9)
11.	$\Box(p \wedge q) \Rightarrow (\Box p \wedge \Box q)$	(Deduction Thrm)

Soundness and Completeness of K

Theorem $\vdash_H \varphi \quad$ iff $\quad \models \varphi$.