Logic and Computation

CS245

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Hilbert System

Agenda

- Axioms
- Examples
- Soundness and Completeness

Formal Deduction

Suppose $\Sigma = \{A_1, A_2, A_3, \ldots\}$. We use the symbol $\vdash$ to denote the relation of *formal deducibility* and write

$$\Sigma \vdash A$$

to mean that A is formally deducible (or *provable* from Σ.

Formal deducibility will be defined by *rules* of formal deduction.

Hilbert System

The *Hilbert System* (H) is an example of a deduction system for the set of propositional logic formulas.

$$Ax_1 : (A \Rightarrow (B \Rightarrow A))$$
$$Ax_2 : (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$
$$Ax_3 : (\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$$
$$MP :$$

$$\frac{A \quad A \Rightarrow B}{B}$$

Hilbert System

(A WFF) A is formally provable by H iff

$$\vdash_H A$$

holds

Example 1

Prove that $\vdash_H (A \Rightarrow A)$ holds

1. $(A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$ (by Ax_1)
2. $(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow (A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))$ (by Ax_2)
3. $(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))$ (by $MP, 1, 2$)
4. $(A \Rightarrow (A \Rightarrow A))$ (by Ax_1)
5. $(A \Rightarrow A)$ (by $MP, 3, 4$)

Example 2

Prove that $\{A \Rightarrow B, B \Rightarrow C\} \vdash_H (A \Rightarrow C)$ holds

1. $(B \Rightarrow C)$	(by Assumption)
2. $((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$	(by Ax_1)
3. $(A \Rightarrow (B \Rightarrow C))$	(by $MP, 1, 2$)
4. $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$	(by Ax_2)
5. $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$	(by $MP, 3, 4$)
6. $(A \Rightarrow B)$	(by Assumption)
7. $(A \Rightarrow C)$	(by $MP, 5, 6$)

Deduction Theorem

$$\Gamma \vdash A \Rightarrow B \quad \text{iff} \quad \Gamma \cup \{A\} \vdash B$$

Example 3

Prove that $\{Ex.2\} \vdash_H (\neg A \Rightarrow (A \Rightarrow B))$ holds

1. $(\neg A \Rightarrow (\neg B \Rightarrow \neg A))$ (by Ax_1)
2. $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ (by Ax_3)
3. $(\neg A \Rightarrow (A \Rightarrow B))$ (by $Ex.2, 1, 2$)

Example 4

Prove: If $\Sigma \vdash_H A$ and $\Sigma \vdash_H (\neg A)$, then $\Sigma \vdash_H B$ for any B.

1. $(\neg A)$	(by Assumption)
2. $((\neg A) \Rightarrow ((\neg B) \Rightarrow (\neg A)))$	(by Ax_1)
3. $((\neg B) \Rightarrow (\neg A))$	$(MP, 1, 2)$
4. $(((\neg B) \Rightarrow (\neg A)) \Rightarrow (A \Rightarrow B))$	(by Ax_3)
5. $(A \Rightarrow B)$	(by $MP3, 4$)
6. A	(by Assumptions)
7. B	(by $MP, 5, 6$)

Example 5

Prove that $\vdash_H (\neg\neg A \Rightarrow A)$

Proof: We show that $\{(\neg\neg A)\} \vdash_H (A))$

1. $(\neg\neg A)$ (by Assumption)
2. $(\neg\neg A) \Rightarrow ((\neg\neg\neg\neg A) \Rightarrow (\neg\neg A))$ (by Ax_1)
3. $((\neg\neg\neg\neg A) \Rightarrow (\neg\neg A))$ (by $MP, 1, 2$)
4. $((\neg\neg\neg\neg A) \Rightarrow (\neg\neg A)) \Rightarrow ((\neg A) \Rightarrow (\neg\neg\neg A)$ (by Ax_3)
5. $(\neg A) \Rightarrow (\neg\neg\neg A)$ (by $MP, 3, 4$)
6. $((\neg A) \Rightarrow (\neg\neg\neg A)) \Rightarrow ((\neg\neg A) \Rightarrow A)$ (by $Ax_3)$
7. $(\neg\neg A) \Rightarrow A$ (by $MP, 5, 6)$
8. A (by $MP, 7, 1$)

Example 6

Prove that $\vdash_H (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

Proof: We show that $\{(A \Rightarrow B)\} \vdash_H (\neg B \Rightarrow \neg A))$

1. $(A \Rightarrow B)$	(by Assumption)
2. $(\neg\neg A) \Rightarrow A$	(by Ex. 6)
3. $(\neg\neg A) \Rightarrow B$	(Ex.2, 1, 2)
4. $B \Rightarrow (\neg\neg B)$	(proof?)
5. $(\neg\neg A) \Rightarrow (\neg\neg B)$	(Ex.2, 3, 4)
6. $((\neg\neg A) \Rightarrow (\neg\neg B)) \Rightarrow ((\neg B) \Rightarrow (\neg A))$	(Ax_3)
7. $((\neg B) \Rightarrow (\neg A))$	($MP, 5, 6$)

Extensions to the Hilbert System

- $(A \Rightarrow (B \Rightarrow (A \wedge B)))$ ($\wedge$ introduction)
- $((A \wedge B) \Rightarrow A), ((A \wedge B) \Rightarrow B)$
- $(A \Rightarrow (A \vee B)), (A \Rightarrow (B \vee A))$ ($\vee$ introduction)
- $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C)))$