Logic and Computation

CS245

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Background

Agenda

- Sets
- Relations
- Functions
- Induction and Recursion

Sets

A *set* is a collection of objects called *members* or *elements*.

We write

$$\alpha \in S$$

to mean that α is a member of S ($\alpha \notin S$ is the opposite).

We write

$$\alpha_1, \ldots, \alpha_n \in S$$

to mean that $\alpha_1 \in S, \ldots,$ and $\alpha_n \in S$.

Sets

Two sets are *equal* (i.e., $S = T$) iff they have the same members:

for every x, $x \in S$ iff $x \in T$.

S is said to be a *subset* of T (i.e., $S \subseteq T$) iff for every x, $x \in S$ implies $x \in T$.

Every set is a subset of itself.

$S = T$ iff $S \subseteq T$ and $T \subseteq S$.

Sets

S is a *proper subset* of T (i.e., $S \subset T$), iff $S \subseteq T$ and $S \neq T$.

Sets are not ordered (e.g., $\{\alpha, \beta\} = \{\beta, \alpha\}$).

Repetition in sets is not important (e.g., $\{\alpha, \alpha, \beta\} = \{\alpha, \beta\}$).

The *empty set* $\emptyset$ has no members. Hence, $\emptyset \subseteq S$ for all S (why?).

Sets

What are the concrete set that represent:

$$\{x \mid x < 100 \text{ and } x \text{ is prime}\}$$

$$\{x \mid x = 0 \text{ or } x = 1 \text{ or } x = 2\}$$

Sets

We define

$\overline{S} = \{x \mid x \notin S\}$ (*complement*)

$S \cup T = \{x \mid x \in S$ or $x \in T\}$ (*union*)

$S \cap T = \{x \mid x \in S$ and $x \in T\}$ (*intersection*)

$S - T = \{x \mid x \in S$ and $x \notin T\}$ (*difference*)

Sets

We define

$$\bigcup_{i \in I} S_i = \{x \mid x \in S_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} S_i = \{x \mid x \in S_i \text{ for all } i \in I\}$$

Relations

The *ordered pair* of objects α and β is written as $\langle \alpha, \beta \rangle$.

Then $\langle \alpha, \beta \rangle = \langle \alpha_1, \beta_1 \rangle$ iff $\alpha = \alpha_1$ and $\beta = \beta_1$.

Similarly, one can define and ordered n-tuple $\langle \alpha_1, \ldots, a_n \rangle$.

One can also define a set of ordered pairs (e.g., $\{\langle m, n \rangle \mid m, n$ are natural numbers and $m < n\}$).

Relations

The *Cartesian product* of sets $S_1, \ldots, S_n$ is defined by

$$S_1 \times \cdots \times S_n = \{\langle x_1, \ldots, x_n \rangle \mid x_1 \in S_1, \ldots, x_n \in S_n\}$$

Let $S^n = \underbrace{S \times \cdots \times S}_{n}$

An n-ary *relation* R on set S is a subset of S^n.

A special binary relation is the equality relation:

$$\{\langle x, y \rangle \mid x, y \in S \text{ and } x = y\}$$

or

$$\{\langle x, x \rangle \mid x \in S\}$$

Relations

For a binary relation R, we often write xRy to denote $\langle x, y\rangle \in R$.

R is *reflexive* on S, iff for any $x \in S$, xRx.

R is *symmetric* on S, iff for any $x, y \in S$, whenever xRy, then yRx.

R is *transitive* on S, iff for any $x, y, z \in S$, whenever xRy and yRz, then xRz.

R is an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Relations

Suppose R is an equivalence relation on S. For any $x \in S$ the set

$$\overline{x} = \{y \in S \mid xRy\}$$

is called the *R-equivalence class of x*.

R-equivalence classes make a *partition* of S.

Functions

A *function (mapping)* f is a set of ordered pairs such that if $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in f$, then $y = z$.

The *domain* $dom(f)$ of f is the set

$$\{x \mid \langle x, y \rangle \in f \text{ for some } y\}$$

The *range* $ran(f)$ of f is the set

$$\{y \mid \langle x, y \rangle \in f \text{ for some } x\}$$

Functions

$f(x)$ denotes the unique element in $y \in ran(f)$, where $x \in dom(f)$ and $\langle x, y \rangle \in f$.

If f is a function with $dom(f) = S$ and $ran(f) \subseteq T$, we say that f is a function from S to T and denote it by

$$f : S \longrightarrow T$$

Similarly, one can define n-ary functions.

Functions

The *restriction* of R to S_1 is the n-ary relation $R \cap S_1^n$.

Suppose $f : S \longrightarrow T$ is a function and $S_1 \subseteq S$. The *restriction* of f to S_1 is the function

$$f \mid S_1 : S_1 \longrightarrow T$$

Functions

A function $f : S \longrightarrow T$ is *onto* if $ran(f) = T$

A function is *one-to-one* if $f(x) = f(y)$ implies $x = y$.

Set Cardinality

Two sets S and T are *equipotent* (i.e., $S \sim T$) iff there is a one-to-one mapping from S onto T.

$\sim$ is an equivalence relations (why?)

A *cardinal* of a set S is denoted by $|S|$ where:

$$|S| = |T| \text{ iff } S \sim T.$$

A set S is to be *countably infinite*, iff $|S| = |N|$. A set S is said to be *countable*, iff $|S| \leq |N|$ (i.e., S is finite or countably infinite).

Set Cardinality

Theorem 1. A subset of a countable set is countable.

Theorem 2. The union of any finite number of countable sets is countable.

Theorem 3. The union of any countably many countable sets is countable.

Set Cardinality

Theorem 4. The Cartesian product of any finite number of countable sets id countable.

Theorem 5. The set of all finite sequences with the members of a countable set as components is countable.

Induction and Proofs

Definition 1. (natural numbers)

[1] $0 \in N$.

[2] For any n, if $n \in N$, then $n' \in N$, where n' is the successor of n.

[3] $n \in N$ only if n has been generated by [1] and [2].

Induction and Proofs

Definition 1. N is the smallest set S such that

[1] $0 \in S$.

[2] For any n, if $n \in S$, then $n' \in S$

Induction and Proofs

Theorem. Suppose R is a unary relation. If

[1] $R(0)$.

[2] For any $n \in N$, if $R(n)$, then $R(n')$.

then $R(n)$ for any $n \in N$.