Algorithms for Fast Linear System Solving and Rank Profile Computation

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Rank profile

Given $A \in \mathsf{K}^{n \times m}$ over a finite field K.

- RankProfile: Compute the rank r and the lexicographically minimal lists $[i_1, i_2, \dots, i_r]$ of row indices of A such that these rows of A are linearly independent.

Pivot locations in the column echelon form:

$$\begin{bmatrix} \circledast & & & & & & & & \\ * & & & & & & & & \\ * & \circledast & & & & & & & \\ * & * & \circledast & & & & & & \\ * & * & * & \circledast & & & & & \\ * & * & * & * & \circledast & & & & \end{bmatrix}$$

Row rank profile: $[1, 3, 4, 5, 6]$

Applications: Gröbner basis computations, computational number theory, etc.

Linear system

- LinSys: Given $b \in \mathsf{K}^{n \times 1}$, compute a particular solution $x \in \mathsf{K}^{m \times 1}$ to $Ax = b$,

Consistent: x can be read off from the last column in the reduced row echelon form.

$$\left[\begin{array}{ccccccccc||c} 1 & * & & & * & * & & * & & x_1 \\ & & 1 & & * & * & & * & & x_3 \\ & & & 1 & * & * & & * & & x_4 \\ & & & & & & 1 & * & & x_7 \\ & & & & & & & & 1 & x_9 \\ & & & & & & & & & \end{array}\right]$$

Solution vector $x^T = \left[\begin{array}{ccccccccc} x_1 & 0 & x_3 & x_4 & 0 & 0 & x_7 & 0 & x_9 \end{array}\right]$

Linear system

- LinSys: Given $b \in \mathsf{K}^{n \times 1}$, compute a particular solution $x \in \mathsf{K}^{m \times 1}$ to $Ax = b$, or a certificate of inconsistency[1]: a row vector $u \in \mathsf{K}^{1 \times n}$ such that $uA = 0$ and $ub \neq 0$.

Inconsistent: u can be read off from the last row of the transformation matrix.

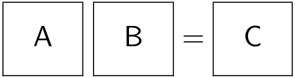
Transform $\left[\; A \;\|\; b \;|\; I \;\right] \Rightarrow \left[\; R \;\|\; * \;|\; U \;\right]$

$$\left[\begin{array}{ccccccccc||c|cccccc}
\circledast & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
 & & \circledast & * & * & * & * & * & * & * & * & * & * & * & * & * \\
 & & & \circledast & * & * & * & * & * & * & * & * & * & * & * & * \\
 & & & & & & \circledast & * & * & * & * & * & * & * & * & * \\
 & & & & & & & & \circledast & * & * & * & * & * & * & * \\
 & & & & & & & & & \circledast & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*}
\end{array}\right]$$

[1]Giesbrecht, Lobo & Saunders (1998)

Notation

- **K :** a finite field.
- **Cost model:** counting scalar field operations of type $\{+, -, \times, /\}$ from K.
- $\boldsymbol{n}$ **:** row dimension of A.
- $\boldsymbol{m}$ **:** column dimension of A.
- $\boldsymbol{r}$ **:** rank of A.
- $\boldsymbol{\omega}$ **:** exponent of matrix multiplication, $2 < \omega \leq 3$. Multiply two $n \times n$ matrices

  in time $O(n^{\omega})$.
- $\boldsymbol{o(1)}$: hides log factors in the cost estimates.

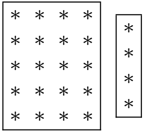
$$O(n \log n \log\log n) = (n)^{1+o(1)}$$

.

Notation

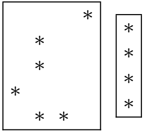
- **$|A|$:** number of nonzero entries of A.

  Dense matrices: $|A| \in \Theta(nm)$

  $cost : O(nm)$

  Sparse matrices: $|A| \in o(nm)$

  $cost : O(|A|)$

  In this talk, we assume $|A| \geq \max(n, m)$.

Notation

- $\boldsymbol{\mu(A)}$ **:** time required to multiply a vector by A in black box approach[2]. It follows a different cost model, in this talk, we have $\mu(A) \in O(|A|)$.

[2] Kaltofen & Saunders (1991)

Previous results for RANK and RANKPROFILE

Deterministic algorithm

- Dumas, Gautier & Pernet (2013); Jeannerod, Pernet & Storjohann (2013)
  - $O(nmr^{\omega-2}) \longleftarrow$ RANKPROFILE

Monte Carlo randomized algorithms

- Kaltofen & Saunders (1991); Chen, Eberly, Kaltofen, Saunders, Turner & Villard (2002)
  - $(r^\omega + nm)^{1+o(1)} \longleftarrow$ RANK
- Wiedemann (1986); Kaltofen & Saunders (1991); Eberly (2003)
  - $(\mu(A)\, r)^{1+o(1)}$ or $(|A|\, r)^{1+o(1)} \longleftarrow$ RANK
- Cheung, Kwok & Lau (2013)
  - $(r^\omega + |A|)^{1+o(1)} \longleftarrow$ RANK
  - computes a list of r linearly independent columns

This talk

- a Monte Carlo algorithm: $(r^\omega + |A|)^{1+o(1)} \longleftarrow$ RANKPROFILE

Previous results for LinSys

Deterministic algorithms

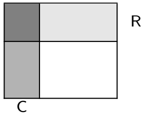
- Dumas, Gautier & Pernet (2013); Jeannerod, Pernet & Storjohann (2013)
  - $O(nmr^{\omega-2})$
- Mulders & Storjohann (2000)
  - $O((n+m)r^2)$

Las Vegas randomized algorithms

- Giesbrecht, Lobo & Saunders (1999); Eberly (2003)
  - $(\mu(A)\,r)^{1+o(1)}$ or $(|A|\,r)^{1+o(1)}$
- Cheung, Kwok & Lau (2013)
  - $(r^\omega + |A|)^{1+o(1)}$

This talk

- a Las Vegas algorithm: $2r^3 + (r^2 + n + m + |R| + |C|)^{1+o(1)}$
- examines at most $r+1$ rows and r columns of A:



Comparison with previous results for LinSys

For a class of input matrices $A \in \mathsf{K}^{n \times n}$ that have

- at most $O(n^{2/3})$ nonzero entries per row and column, and
- $r \in O(n^{1/3})$.

Black box approach:

- $O(n)$ additional space
- $(n^2)^{1+o(1)}$ time

Cheung, Kwok & Lau (2013):

- $O(n^{5/3})$ additional space
- $(n^{5/3})^{1+o(1)}$ time

Our approach:

- $O(n)$ additional space
- $(n)^{1+o(1)}$ time

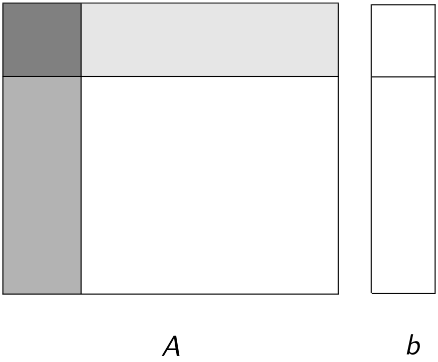
Outline

- Oracle linear solving [Mulders & Storjohann (2000)]
  - application to RankProfile
- Linear independence oracles
  - application to LinSys
  - application to RankProfile
- A relaxed algorithm for online matrix inversion
  - application to RankProfile

Oracle linear solving

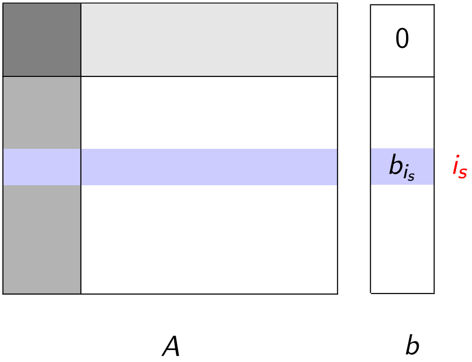
Mulders & Storjohann (2000)

At stage s



Oracle linear solving
Mulders & Storjohann (2000)

At stage s



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Mulders & Storjohann (2000)

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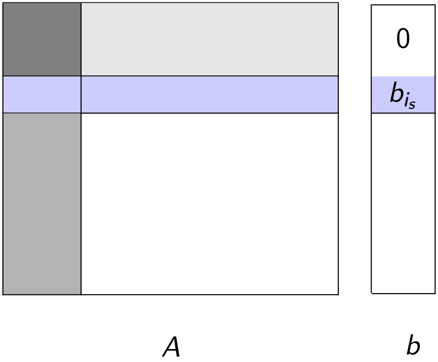

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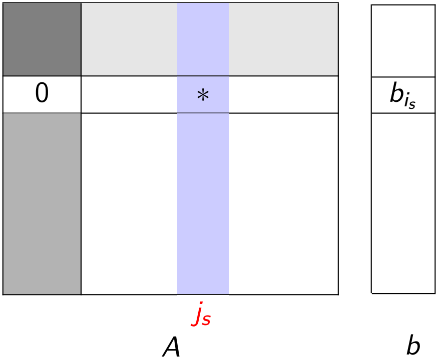

Oracle linear solving
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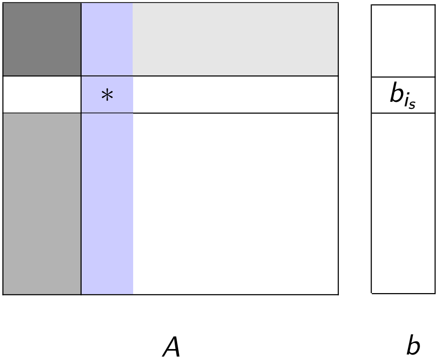
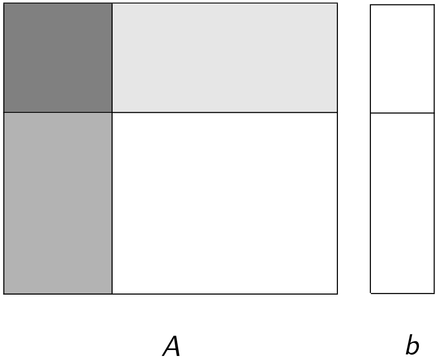


Oracle linear solving

Mulders & Storjohann (2000)

Repeat for stage $s + 1$



Oracle linear solving
Mulders & Storjohann (2000)

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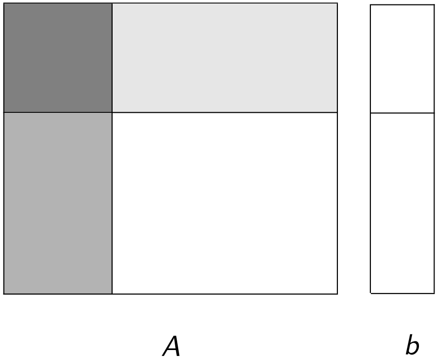

Terminates with $s \leq r$, overall cost $O((n+m)r^2)$ for LinSys.

Contribution 1: Randomized rank profiles

- If b is chosen uniformly and randomly sampled from the column space of A, that is,
  - choose a $w \in \mathsf{K}^{m \times 1}$ uniformly and randomly, and compute $b = Aw$,

  then $[i_1, i_2, \ldots, i_s]$ is the row rank profile of A with probability at least $(1 - 1/\#\mathsf{K})^r$.
- This gives a Monte Carlo algorithm for RankProfile in time $O((n + m)r^2)$.
- The ideas of our improved algorithms for LinSys and RankProfile are the same, except that b is given explicitly for LinSys.
- We focus on algorithms for RankProfile in this presentation.

Goals

Starting complexity: $O((n+m)r^2)$

Goals

1. Decouple the cubic part of the time complexity:

$$2r^3 + (r^2 + nm)^{1+o(1)}$$

2. Exploit possible sparsity of A:

$$2r^3 + (r^2 + |A|)^{1+o(1)}$$

3. Incorporate fast matrix multiplication:

$$(r^\omega + |A|)^{1+o(1)}$$

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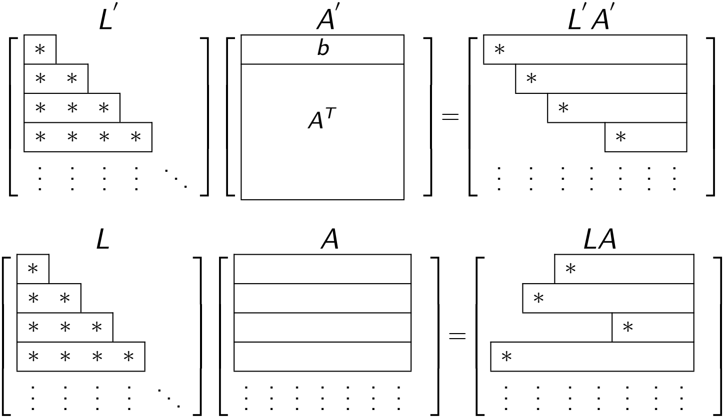
Linear independence oracles

After some simplifications, the steps in the oracle solver algorithm to find i_s and j_s "boil down" to the problem of finding the pivot locations in $L^{'}A^{'}$ and LA. $L^{'}$ and L are coming from Gaussian elimination.



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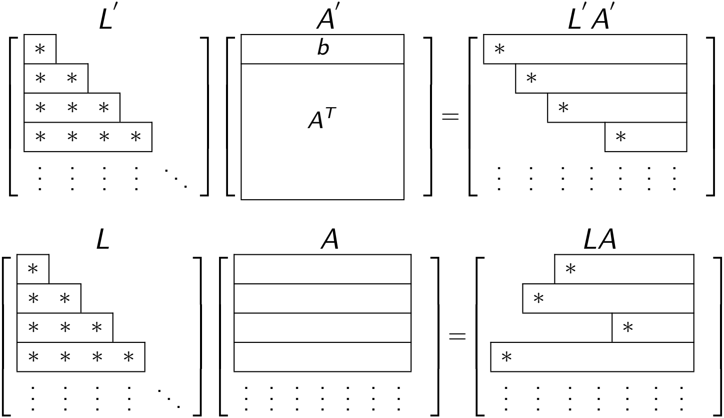
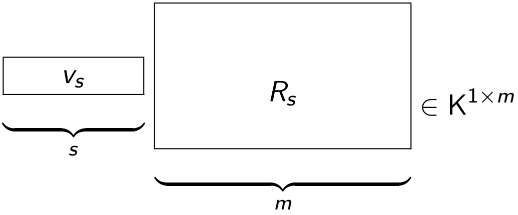


Computing $L^{'}A^{'}$ and LA directly are expensive.

Linear independence oracles

At each stage s, finding j_s is equivalent to finding the first nonzero entry of



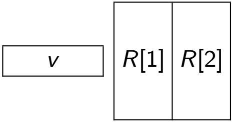
Computing $v_s R_s$ explicitly: $O(sm)$ field operations.

Use *linear independence oracle*: $O(s \log m)$ field operations.

Linear independence oracles

Example. $v \in \mathrm{K}^{1\times s}$, $R \in \mathrm{K}^{s\times 2}$.

- Require 2 dot products to determine if $vR = 0$.



- Idea: take random linear combination of columns of R. Choose $\alpha \in \mathrm{K}$ uniformly and randomly and compute

$$R_{1\sim 2} = R[1] + \alpha R[2]$$

  Require 1 dot product to determine if $vR = 0$ with high probability.

  - $vR_{1\sim 2} \neq 0 \implies vR \neq 0$.
  - If α well chosen, $vR_{1\sim 2} = 0 \implies vR = 0$.
  - α is good with probability $(1 - 1/\#\mathrm{K})$.

Linear independence oracles

T is a *linear independence oracle* for R based on $\alpha_1, \ldots, \alpha_{m-1}$.

$\alpha_1, \ldots, \alpha_{m-1} \in$ K be chosen uniformly and randomly.

$R_{a\sim b}$: a vector that is a linear combination of $R[a], R[a+1], \ldots, R[b]$.

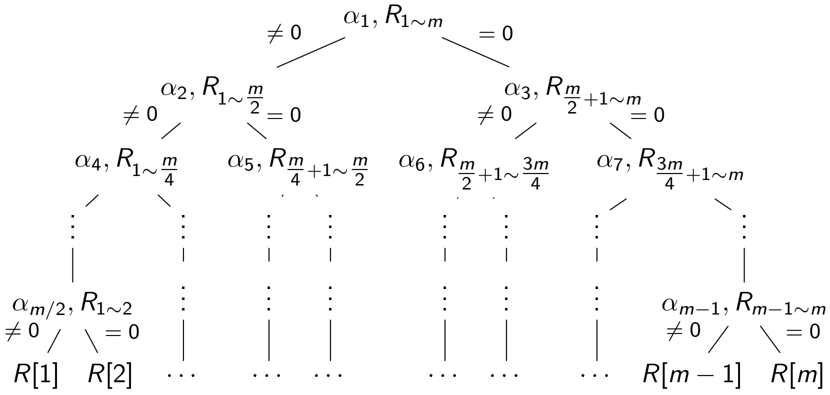


Figure : Oracle tree T for columns of R

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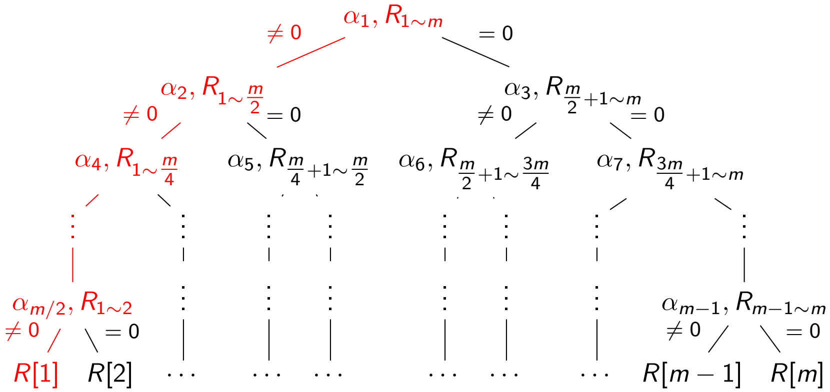

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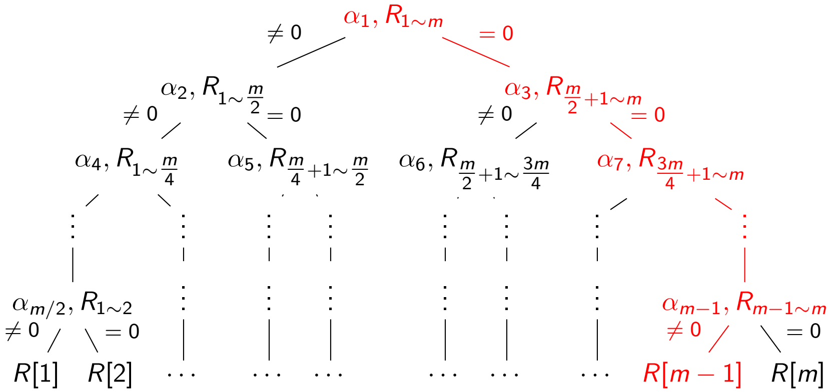


Figure : Oracle tree T for columns of R

Linear independence oracle

Cost

- The first nonzero entry in $v_s R_s$ can be found in $O(s \log m)$ field operations from K.

Probability of correctness

- T_s is correct with respect to v_s with probability at least

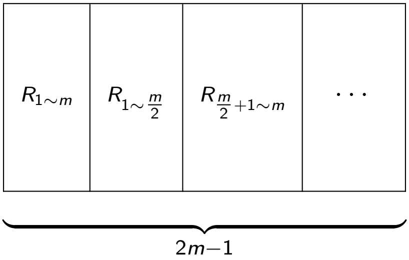
$$(1 - 1/\#\mathrm{K})^{\log_2 m}.$$

- $(T_s)_{1 \le s \le r}$, all based on the same $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$, are correct with respect to $(v_s)_{1 \le s \le r}$ with probability at least

$$(1 - r/\#\mathrm{K})^{\log_2 m}.$$

Linear independence oracles

Data structure for T

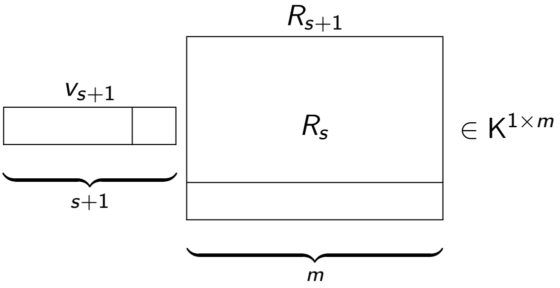


Linear independence oracles

Online construction

Construct T_s for $s = 0, 1, \ldots, r$ in succession.

At stage $s + 1$



Construct T_s for $s = 0, 1, \ldots, r$ in succession.

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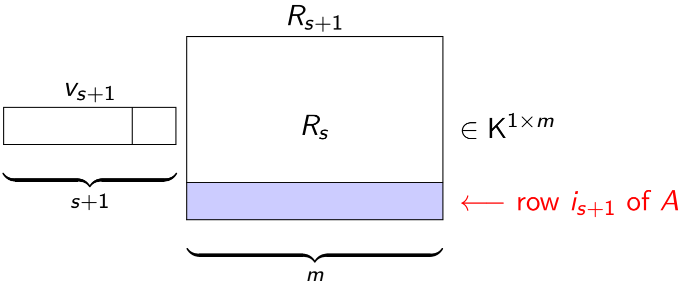


Linear independence oracles

Online construction

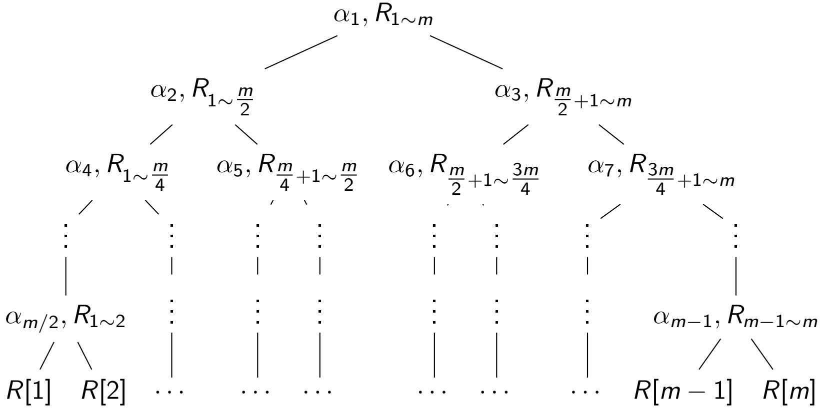
Data structure for T_{s+1}

$R_{1\sim m}$	$R_{1\sim \frac{m}{2}}$	$R_{\frac{m}{2}+1\sim m}$	$\cdots$
			$\cdots$

$2m-1$

$\alpha_1, \ldots, \alpha_{m-1} \in \mathsf{K}$ be chosen uniformly and randomly.

T_{s+1}: Append the $(s+1)^{th}$ row to T_s in a bottom up fashion.



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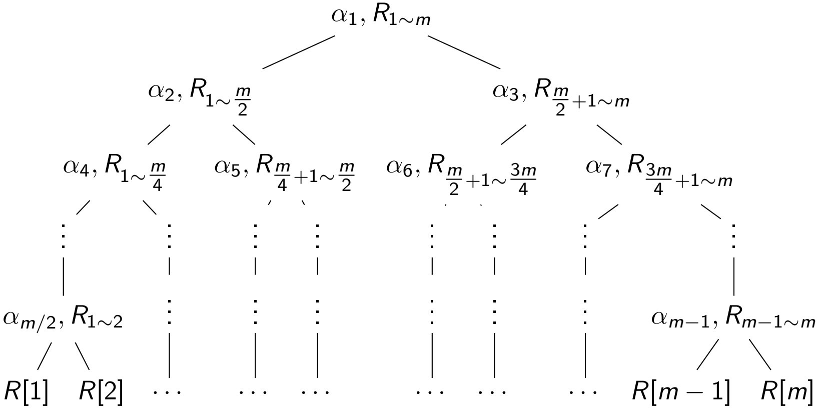


Cost: $O(m)$

Rank Profile

Theorem

There exists a randomized algorithm for RANKPROFILE *that has:*

1. $n + 2m - 2$ *random choices from* K *are required.*
2. *Probability of correctness at least*

$$\left(1 - \frac{1}{\#\mathsf{K}}\right)^r \left(1 - \frac{r}{\#\mathsf{K}}\right)^{\lceil \log_2 n \rceil + \lceil \log_2 m \rceil}$$

3. *The running time is bounded by*

$$\underbrace{2r^3}_{\textit{Inverse}} + O\left(\underbrace{nm}_{b=Aw} + \underbrace{r^2(\log n + \log m)}_{\textit{Use LIOs}} + \underbrace{(n+m)r}_{\textit{Build LIOs}}\right)$$

field operations in K.

Goals

Starting complexity: $O((n+m)r^2)$

Goals

1. Decouple the cubic part of the time complexity:

$$2r^3 + (r^2 + nm)^{1+o(1)}$$

2. Exploit possible sparsity of A:

$$2r^3 + (r^2 + |A|)^{1+o(1)}$$

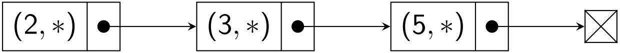
3. Incorporate fast matrix multiplication:

$$(r^\omega + |A|)^{1+o(1)}$$

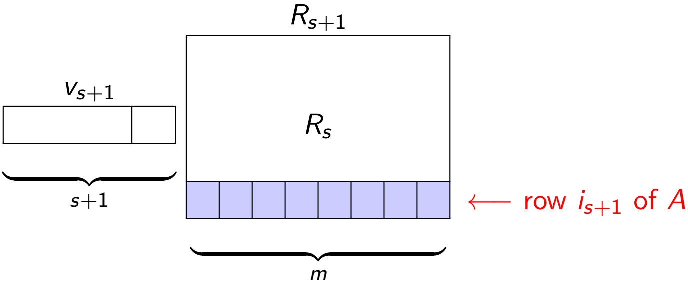
Exploit possible sparsity of A

Use a sparse representation for T.

Example. $R_{a\sim b} = \begin{bmatrix} & * & * & & * & \end{bmatrix}^T$ is represented as



Recall the construction of T_{s+1}



Only nonzero elements of row i_{s+1} of A modifies the associated vectors in T_{s+1}.

Exploit possible sparsity of A

Example. $r = 3$ and $m = 8$ Stage 0

$R_0 = \emptyset$

Cost: $O(m)$

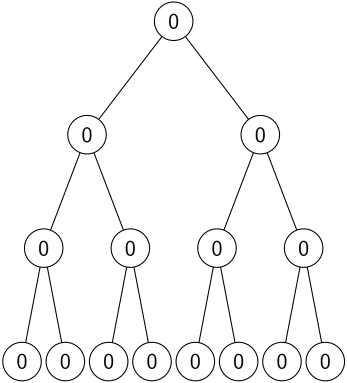
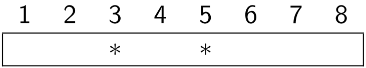


Figure : T_0

Exploit possible sparsity of A

Example. $r = 3$ and $m = 8$ Stage 1



R_1

Cost: $O(2 \log m)$

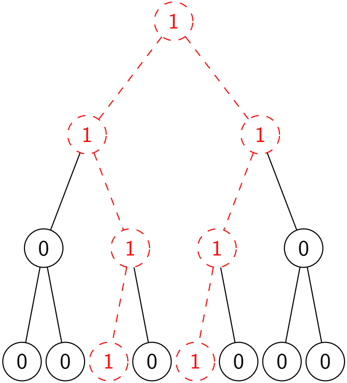


Figure : T_1

Exploit possible sparsity of A

Example. $r = 3$ and $m = 8$ Stage 2

1	2	3	4	5	6	7	8
		*		*			
			*	*			

R_2

Cost: $O(2 \log m)$

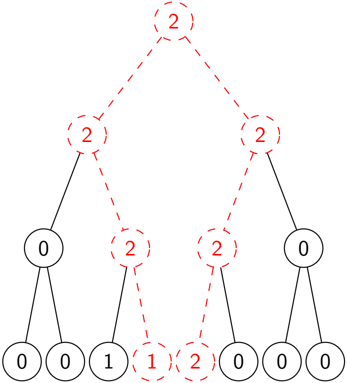


Figure : T_2

Exploit possible sparsity of A

Example. $r = 3$ and $m = 8$ Stage 3

1	2	3	4	5	6	7	8
		*		*			
			*	*			
	*			*			*

R_2

Cost: $O(3 \log m)$

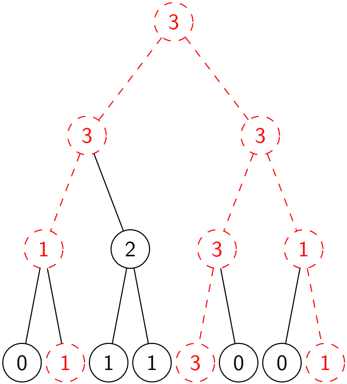


Figure : T_3

Exploit possible sparsity of A

Example. $r = 3$ and $m = 8$ Stage 3

1	2	3	4	5	6	7	8
		*		*			
			*	*			
	*			*			*

R_2

Cost: $O(3 \log m)$

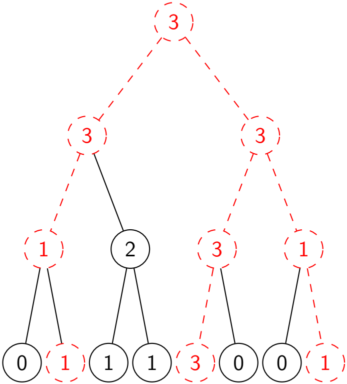


Figure : T_3

Overall cost: $O(m + |R| \log m)$

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field operations in K.

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Goals

1. Decouple the cubic part of the time complexity:

$$2r^3 + (r^2 + nm)^{1+o(1)}$$

2. Exploit possible sparsity of A:

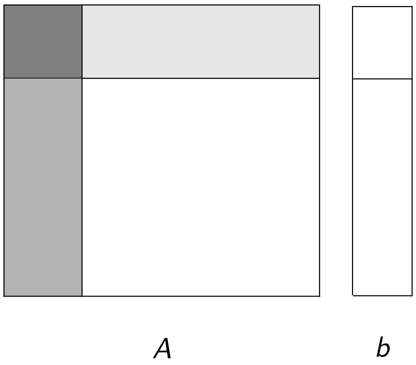
$$2r^3 + (r^2 + |A|)^{1+o(1)}$$

3. Incorporate fast matrix multiplication:

$$(r^\omega + |A|)^{1+o(1)}$$

Incorporate fast matrix multiplication

The leading term $2r^3$ arises from computing the inverse of the leading $s \times s$ submatrix for $s = 1, 2, \ldots, r$.



Incorporate fast matrix multiplication

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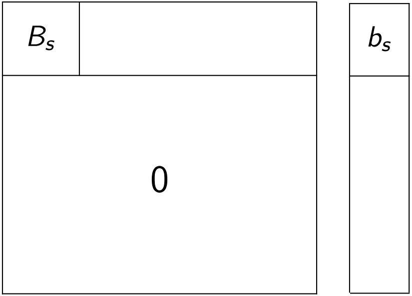
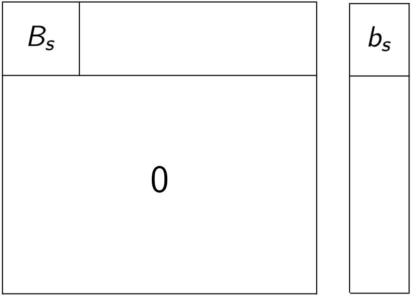


Work matrix B

b

Incorporate fast matrix multiplication

The leading term $2r^3$ arises from computing the inverse of the leading $s \times s$ submatrix for $s = 1, 2, \ldots, r$.



Work matrix B $\qquad$ b

These inverses are used to compute a sequence of subsystem solutions $B_s^{-1} b_s$ for $s = 1, 2, \ldots, r$.

Full inverse decomposition

We give a unique decomposition for the inverse

$$B_s^{-1} = (R_s L_s) \cdots (R_2 L_2)(R_1 L_1)$$

Example. $B_6^{-1} =$

$$\underbrace{\begin{bmatrix} 1 &&&&& * \\ & 1 &&&& * \\ && 1 &&& * \\ &&& 1 && * \\ &&&& 1 & * \\ &&&&& * \end{bmatrix}}_{R_6} \underbrace{\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ * & * & * & * & * & 1 \end{bmatrix}}_{L_6} \underbrace{\begin{bmatrix} 1 &&&& * & \\ & 1 &&& * & \\ && 1 && * & \\ &&& 1 & * & \\ &&&& * & \\ &&&&& 1 \end{bmatrix}}_{R_5} \underbrace{\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ * & * & * & * & 1 & \\ &&&&& 1 \end{bmatrix}}_{L_5} \underbrace{\begin{bmatrix} 1 &&& * && \\ & 1 && * && \\ && 1 & * && \\ &&& * && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}}_{R_4} \underbrace{\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ * & * & * & 1 && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}}_{L_4} \cdots \underbrace{\begin{bmatrix} * &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}}_{R_1} \underbrace{\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}}_{L_1}$$

To compute a sequence of subsystem solutions $B_s^{-1} b_s$ for $s = 1, 2, \ldots, r$, it is sufficient to solve the following problem.

- OnlineInverse: Suppose the rows of $B \in \mathsf{K}^{r \times r}$ with generic rank profile are given one at a time, from first to last. As soon as rows $1, 2, \ldots, r$ of B are given, the pair of matrices (R_s, L_s) should be produced, for $s = 1, 2, \ldots, r$.

Iterative algorithm for OnlineInverse: $2r^3 + O(r^2)$

Full inverse decomposition

We give a unique decomposition for the inverse

$$B_s^{-1} = (R_s L_s) \cdots (R_2 L_2)(R_1 L_1)$$

Example. $B_6^{-1} =$

$$\underset{R_6}{\begin{bmatrix} 1 & & & & & * \\ & 1 & & & & * \\ & & 1 & & & * \\ & & & 1 & & * \\ & & & & 1 & * \\ & & & & & * \end{bmatrix}} \underset{L_6}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ * & * & * & * & * & 1 \end{bmatrix}} \underset{R_5}{\begin{bmatrix} 1 & & & & * & \\ & 1 & & & * & \\ & & 1 & & * & \\ & & & 1 & * & \\ & & & & * & \\ & & & & & 1 \end{bmatrix}} \underset{L_5}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ * & * & * & * & 1 & \\ & & & & & 1 \end{bmatrix}} \underset{R_4}{\begin{bmatrix} 1 & & & * & & \\ & 1 & & * & & \\ & & 1 & * & & \\ & & & * & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}} \underset{L_4}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ * & * & * & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}} \cdots \underset{R_1}{\begin{bmatrix} * & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}} \underset{L_1}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}}$$

To compute a sequence of subsystem solutions $B_s^{-1} b_s$ for $s = 1, 2, \ldots, r$, it is sufficient to solve the following problem.

- OnlineInverse: Suppose the rows of $B \in \mathsf{K}^{r \times r}$ with generic rank profile are given one at a time, from first to last. As soon as rows $1, 2, \ldots, r$ of B are given, the pair of matrices (R_s, L_s) should be produced, for $s = 1, 2, \ldots, r$.

Iterative algorithm for OnlineInverse: $2r^3 + O(r^2)$

How to incorporate matrix multiplication?

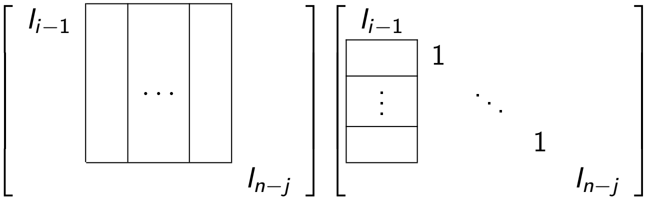
Relax, but anticipate

We adopt two ideas used in relaxed and online algorithms.

1. **Use a *relaxed* representation for B_s^{-1}.**

   Key observation: $(R_j L_j)(R_{j-1} L_{j-1}) \cdots (R_i L_i)$ can be expressed as

$R_{j\sim i} L_{j\sim i} =$



Relax, but anticipate

Represent B_s^{-1} as the product of $\mathrm{HammingWeight}(s) \leq \lceil \log s \rceil$ pair of structured matrices.

Example. The relaxed representation of B_s^{-1} for $1 \leq s \leq 8$.

s	**Relaxed representation of B_s^{-1}**
$1 = (1)_2$	$(R_{1\sim 1} L_{1\sim 1})$
$2 = (10)_2$	$(R_{2\sim 1} L_{2\sim 1})$
$3 = (11)_2$	$(R_{3\sim 3} L_{3\sim 3})(R_{2\sim 1} L_{2\sim 1})$
$4 = (100)_2$	$(R_{4\sim 1} L_{4\sim 1})$
$5 = (101)_2$	$(R_{5\sim 5} L_{5\sim 5})(R_{4\sim 1} L_{4\sim 1})$
$6 = (110)_2$	$(R_{6\sim 5} L_{6\sim 5})(R_{4\sim 1} L_{4\sim 1})$
$7 = (111)_2$	$(R_{7\sim 7} L_{7\sim 7})(R_{6\sim 5} L_{6\sim 5})(R_{4\sim 1} L_{4\sim 1})$
$8 = (1000)_2$	$(R_{8\sim 1} L_{8\sim 1})$

Example.

$$B_6^{-1} = \overset{R_{6\sim5}}{\begin{bmatrix} 1 & & & & * & * \\ & 1 & & & * & * \\ & & 1 & & * & * \\ & & & 1 & * & * \\ & & & & * & * \\ & & & & * & * \end{bmatrix}} \overset{L_{6\sim5}}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ * & * & * & * & 1 & \\ * & * & * & * & & 1 \end{bmatrix}} \overset{R_{4\sim1}}{\begin{bmatrix} * & * & * & * & & \\ * & * & * & * & & \\ * & * & * & * & & \\ * & * & * & * & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}} \overset{L_{4\sim1}}{\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}}$$

$$B_7^{-1} = \overset{R_7}{\begin{bmatrix} 1 & & & & & & * \\ & 1 & & & & & * \\ & & 1 & & & & * \\ & & & 1 & & & * \\ & & & & 1 & & * \\ & & & & & 1 & * \\ & & & & & & * \end{bmatrix}} \overset{L_7}{\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ * & * & * & * & * & * & 1 \end{bmatrix}} \overset{R_{6\sim5}}{\begin{bmatrix} 1 & & & & * & * & \\ & 1 & & & * & * & \\ & & 1 & & * & * & \\ & & & 1 & * & * & \\ & & & & * & * & \\ & & & & * & * & \\ & & & & & & 1 \end{bmatrix}} \overset{L_{6\sim5}}{\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ * & * & * & * & 1 & & \\ * & * & * & * & & 1 & \\ & & & & & & 1 \end{bmatrix}} \overset{R_{4\sim1}}{\begin{bmatrix} * & * & * & * & & & \\ * & * & * & * & & & \\ * & * & * & * & & & \\ * & * & * & * & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}} \overset{L_{4\sim1}}{\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}}$$

$$B_8^{-1} = \overset{R_{8\sim1}}{\begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}} \overset{L_{8\sim1}}{\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}}$$

Relax, but anticipate

The relaxed representation is constructed in an incremental fashion.

Example.

$$\begin{aligned} B_8^{-1} &= (R_8 L_8)\ B_7^{-1} \\ &= (R_8 L_8)\ (R_7 L_7)\ (R_{6\sim5} L_{6\sim5})\ (R_{4\sim1} L_{4\sim1}) \\ &= (R_{8\sim7} L_{8\sim7})\ (R_{6\sim5} L_{6\sim5})\ (R_{4\sim1} L_{4\sim1}) \\ &= (R_{8\sim5} L_{8\sim5})\ (R_{4\sim1} L_{4\sim1}) \\ &= (R_{8\sim1} L_{8\sim1}) \end{aligned}$$

Relax, but anticipate

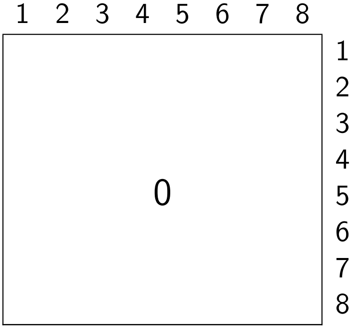
2. ***Anticipate* computations**.

   To compute the pair (R_s, L_s), we need to apply the inverse B_{s-1}^{-1} to column s of B.

   At stage $s-1$, apply parts of our representation for B_{s-1}^{-1} to multiple columns of B such that column s of B have been premultiplied with B_{s-1}^{-1} at the beginning of stage s.

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 1

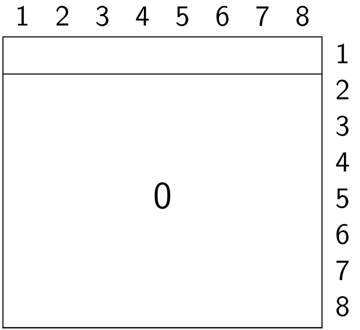


B

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 1

- The first row of B is given.



B

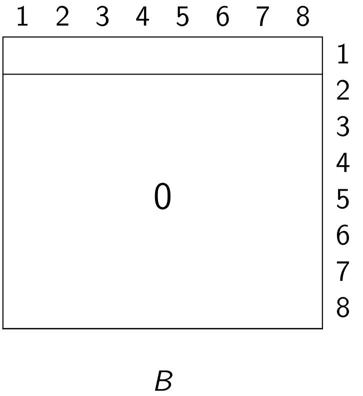
Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 1

- The first row of B is given.
- Compute $(R_1 L_1)$ of shape:

$$[*]\,[1]$$



Relax, but anticipate

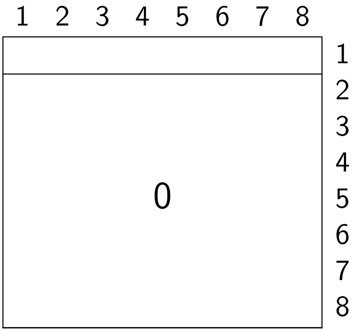
Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 1

- The first row of B is given.
- Compute $(R_1 L_1)$ of shape:

$$[*]\,[1]$$

- $B_1^{-1} = (R_1 L_1)$.



B

Relax, but anticipate

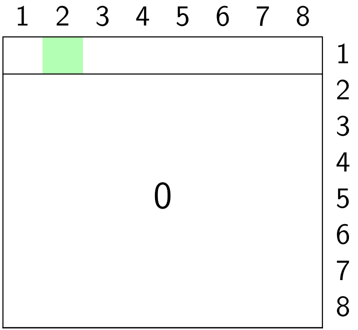
Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 1

- The first row of B is given.
- Compute $(R_1 L_1)$ of shape:

$$[*]\,[1]$$

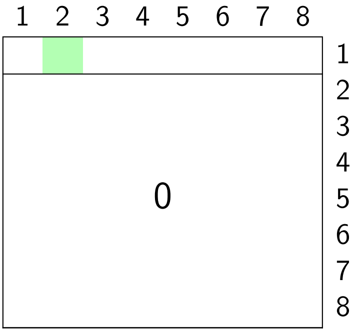
- $B_1^{-1} = (R_1 L_1)$.
- Apply $R_1 L_1$ to column 2 of B.



B

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 2

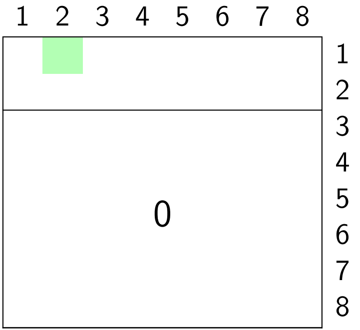


B

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 2

- The 2^{nd} row of B is given.



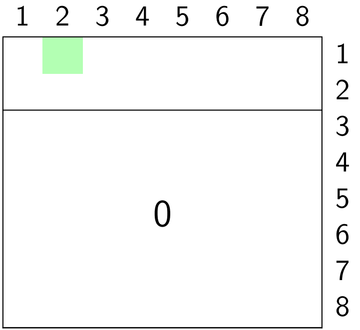
B

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 2

- The 2^{nd} row of B is given.
- Compute $(R_2 L_2)$ of shape:

$$\begin{bmatrix}1 & * \\ & *\end{bmatrix}\begin{bmatrix}1 & \\ * & 1\end{bmatrix}$$



B

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

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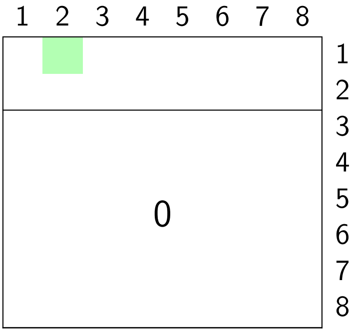
- Compute $(R_2 L_2)$ of shape:

$$\begin{bmatrix} 1 & * \\ & * \end{bmatrix} \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$$

- Compress $(R_2 L_2)(R_1 L_1) = (R_{2\sim 1} L_{2\sim 1})$

$$\begin{bmatrix} 1 & * \\ & * \end{bmatrix} \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix} \begin{bmatrix} * & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$



B

Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

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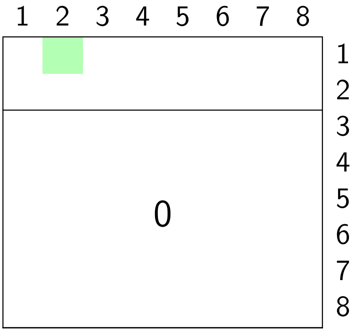
$$\begin{bmatrix} 1 & * \\ & * \end{bmatrix} \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix} \begin{bmatrix} * & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

- $B_2^{-1} = (R_{2\sim 1} L_{2\sim 1})$.



B

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 2

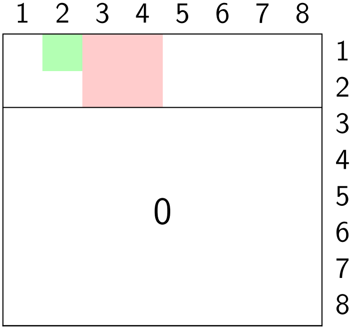
- The 2^{nd} row of B is given.
- Compute $(R_2 L_2)$ of shape:

$$\begin{bmatrix} 1 & * \\ & * \end{bmatrix} \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$$

- Compress $(R_2 L_2)(R_1 L_1) = (R_{2\sim 1} L_{2\sim 1})$

$$\begin{bmatrix} 1 & * \\ & * \end{bmatrix} \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix} \begin{bmatrix} * & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

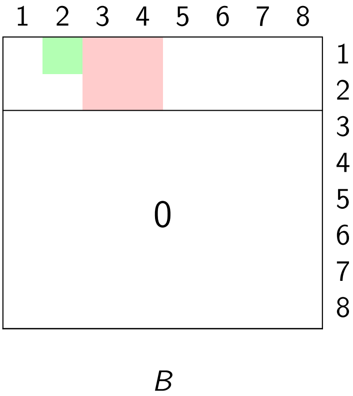
- $B_2^{-1} = (R_{2\sim 1} L_{2\sim 1})$.
- Apply $(R_{2\sim 1} L_{2\sim 1})$ to columns 3,4 of B.



B

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

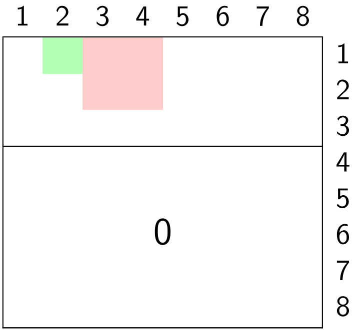
Stage 3



Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 3

- The 3^{rd} row of B is given.



B

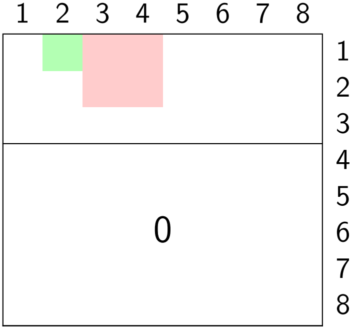
Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 3

- The 3^{rd} row of B is given.
- Compute $(R_3 L_3)$ of shape:

$$\begin{bmatrix} 1 & & * \\ & 1 & * \\ & & * \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{bmatrix}$$



B

Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

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- The 3^{rd} row of B is given.
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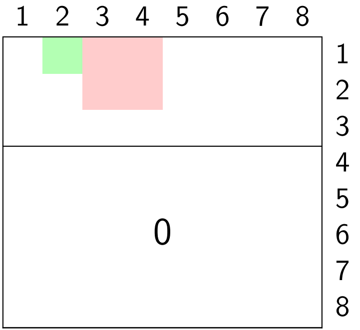
- $B_3^{-1} = (R_3 L_3)(R_{2\sim 1} L_{2\sim 1})$.



B

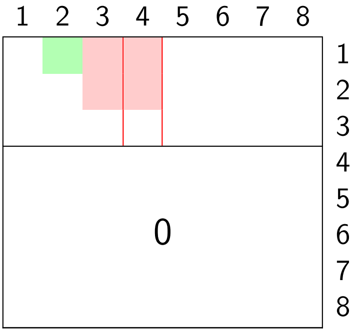
Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 3

- The 3^{rd} row of B is given.
- Compute $(R_3 L_3)$ of shape:

$$\begin{bmatrix} 1 & & * \\ & 1 & * \\ & & * \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{bmatrix}$$

- $B_3^{-1} = (R_3 L_3)(R_{2\sim 1} L_{2\sim 1})$.
- Apply $(R_3 L_3)$ to column 4 of B.

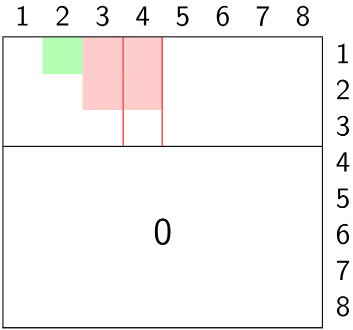


B

Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 4

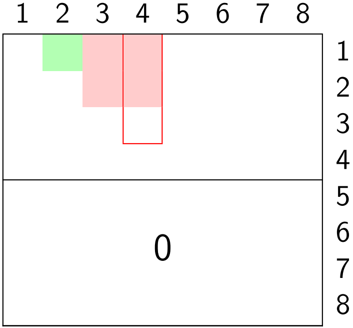


B

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 4

- The 4^{th} row of B is given.



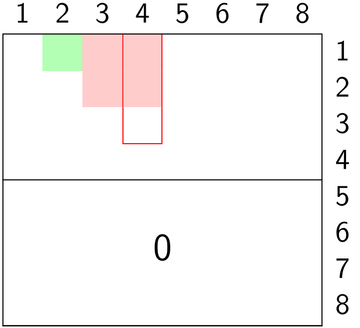
B

Example. The computations of the first four stages for $B \in \mathsf{K}^{8\times 8}$.

Stage 4

- The 4^{th} row of B is given.
- Compute $(R_4 L_4)$ of shape:

$$\begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{bmatrix}$$



B

Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 4

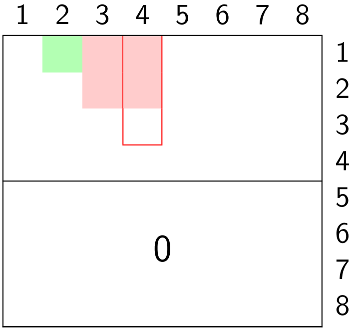
- The 4^{th} row of B is given.
- Compute $(R_4 L_4)$ of shape:

$$\begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{bmatrix}$$

- Compress $(R_4 L_4)(R_3 L_3) = (R_{4\sim 3} L_{4\sim 3})$

$$\begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} 1 & & * & \\ & 1 & * & \\ & & * & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{bmatrix}$$



B

Relax, but anticipate

Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 4

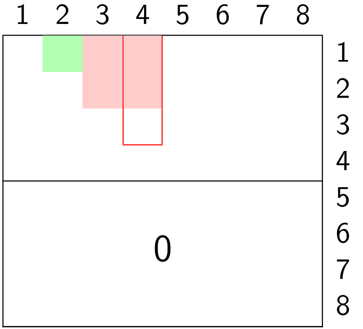
- The 4^{th} row of B is given.
- Compute $(R_4 L_4)$ of shape:

$$\begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{bmatrix}$$

- Compress
$(R_{4\sim 3} L_{4\sim 3})(R_{2\sim 1} L_{2\sim 1}) = (R_{4\sim 1} L_{4\sim 1})$

$$\begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ * & * & 1 & \\ * & * & & 1 \end{bmatrix} \begin{bmatrix} * & * & & \\ * & * & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$



B

Relax, but anticipate

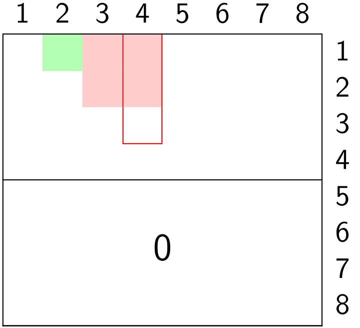
Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 4

- The 4^{th} row of B is given.
- Compute $(R_4 L_4)$ of shape:

$$\begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{bmatrix}$$

- $B_4^{-1} = (R_{4\sim 1} L_{4\sim 1})$.



B

Relax, but anticipate

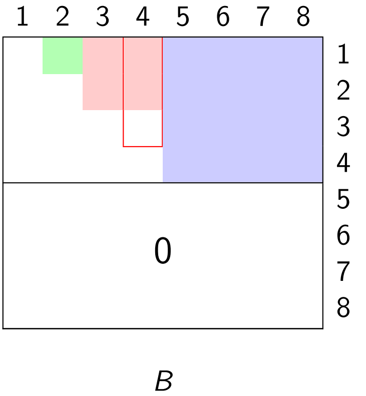
Example. The computations of the first four stages for $B \in \mathrm{K}^{8\times 8}$.

Stage 4

- The 4^{th} row of B is given.
- Compute $(R_4 L_4)$ of shape:

$$\begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & 1 \end{bmatrix}$$

- $B_4^{-1} = (R_{4\sim 1} L_{4\sim 1})$.
- Apply $R_{4\sim 1} L_{4\sim 1}$ to columns 5,6,7,8 of B.



Relax, but anticipate

- At stages $2^k = 1, 2, 4, \ldots$ the explicit inverse has been computed.
- The number of compressions done at stage s is equal to the maximal $c \in \mathbb{Z}$ such that $2^c \mid s$, thus some stages are more costly than others.
- By taking into account the special structure of the $(R_{j\sim i} L_{j\sim i})$ matrices, an amortized analysis of the above approach yields an algorithm for OnlineInverse with overall running time bounded by $O(r^\omega)$ field operations from K.

Rank profile algorithm

Steps:

- Use Cheung, Kwok & Lau's (2013) algorithm to find a subset of r linearly independent columns of A (with high probability).
- Use a Toeplitz preconditioner L such that the leading $s \times s$ submatrix of BL, $1 \leq s \leq r$, are nonsingular.
- Incorporate the relaxed approach for OnlineInverse into the oracle rank profile algorithm.

Overall running time: $(r^{\omega} + n + m + |A|)^{1+o(1)}$

Future work

- Our algorithm for LinSys has overall running time

$$2r^3 + (r^2 + n + m + |R| + |C|)^{1+o(1)},$$

an open problem is to reduce the leading term $2r^3$ to $O(r^\omega)$ by incorporating fast matrix multiplication.

- Find other applications for our relaxed algorithm for online matrix inversion.