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Structured Preconditioners for Smith Form Computations

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MMath Research Project, 2026

Outline

Smith Normal Form

Definition

Let $A \in \mathbf{R}^{n \times n}$ nonsingular. Then there exist (not necessarily unique) unimodular matrices $U, V \in \mathbf{R}^{n \times n}$ such that

$$A = USV, \qquad S = \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix}$$

with $s_i \mid s_{i+1}$ for all $1 \leq i < n$. The matrix S is called the *Smith normal form* of A, which we write as $S = \mathrm{SNF}(A)$.

Note that the standard definition has $UAV = S$.

Example

Take the matrix over $\mathbf{R} = \mathbb{Z}$ given by

$$A_{\text{ex1}} = \begin{bmatrix} 18 & 15 \\ 6 & 3 \end{bmatrix}. \text{ Then } \qquad \mathrm{SNF}(A_{\text{ex1}}) = \begin{bmatrix} 3 & \\ & 12 \end{bmatrix}$$

Determinantal Divisors

An important feature of the entries of the Smith normal form:

Remark

For $i = 1, \ldots, n$, the product

$$s_i^* = s_1 s_2 \cdots s_i,$$

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Hermite Normal Form

Definition

For $A \in \mathbf{R}^{n \times n}$ nonsingular, there exists a (unique) unimodular $W \in \mathbf{R}^{n \times n}$ with

$$A = WH, \qquad H = \begin{bmatrix} h_1 & * & \cdots & * \\ & h_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & h_n \end{bmatrix}$$

where the entries above the diagonal entry h_i are reduced modulo h_i.

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Example

Take the matrix over $\mathbf{R} = \mathbb{Z}$ given by

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This can be seen in the example from the previous slide.

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Triangular Smith Form

The Hermite form is historically easier to compute than the Smith form, requiring only unimodular row operations, which motivated the following definition.

Definition (Villard, 1995)

If a nonsingular matrix $H \in \mathbf{R}^{n \times n}$ in Hermite normal form has $h_i = s_i$ for all $i = 1, \ldots, n$, then H is said to be in *triangular Smith form*.

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Example

Take the matrix over $\mathbb{Z}$ given by

$$A_{\text{ex2}} = \begin{bmatrix} 33 & 15 \\ 9 & 3 \end{bmatrix}. \text{ Then } \operatorname{HNF}(A_{\text{ex2}}) = \begin{bmatrix} 3 & 9 \\ & 12 \end{bmatrix}, \operatorname{SNF}(A_{\text{ex2}}) = \begin{bmatrix} 3 & \\ & 12 \end{bmatrix}.$$

We have been using an arbitrary PID $\mathbf{R}$, but we are interested mainly in the case where $\mathbf{R} = \mathbf{K}[x]$, with $\mathbf{K}$ a field. The concept of GCDs, and therefore Hermite form and Smith form, extend to polynomials over a field.

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Example

Take the matrix over $\mathbf{R} = \mathbb{Z}_5[x]$ given by

$$A_{\text{ex3}} = \begin{bmatrix} x(x+3) & x(2x+1) \\ x(x+3) & x \end{bmatrix}.$$

Then we have

$$\text{HNF}(A_{\text{ex3}}) = \begin{bmatrix} x^2+3x & x \\ & x^2 \end{bmatrix}, \quad \text{SNF}(A_{\text{ex3}}) = \begin{bmatrix} x & \\ & x^3+3x^2 \end{bmatrix}.$$

Preconditioning for Triangular Smith Form

Preconditioning a matrix to be in triangular Smith form means that calculating the Smith form is reduced to calculating the Hermite form.

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Essentially: We can reduce Smith form to Hermite form by multiplying by a **dense** unit lower triangular matrix.

Example

Recall the matrices A_{ex1} and A_{ex2} from the previous examples, given by

$$A_{\text{ex1}} = \begin{bmatrix} 18 & 15 \\ 6 & 3 \end{bmatrix} \qquad \text{HNF}(A_{\text{ex1}}) = \begin{bmatrix} 6 & 3 \\ & 6 \end{bmatrix}, \quad \text{SNF}(A_{\text{ex1}}) = \begin{bmatrix} 3 & \\ & 12 \end{bmatrix}.$$

$$A_{\text{ex2}} = \begin{bmatrix} 33 & 15 \\ 9 & 3 \end{bmatrix} \qquad \text{HNF}(A_{\text{ex2}}) = \begin{bmatrix} 3 & 9 \\ & 12 \end{bmatrix}, \text{SNF}(A_{\text{ex2}}) = \begin{bmatrix} 3 & \\ & 12 \end{bmatrix}.$$

We can see that

$$A_{\text{ex1}} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = A_{\text{ex2}}.$$

So this matrix preconditions A_{ex1} in this way.

We wish to use structured matrices — for example, a *Toeplitz* matrix.

Dense Matrix:

$$\begin{bmatrix} 1 & & & & & & \\ * & 1 & & & & & \\ * & * & 1 & & & & \\ * & * & * & 1 & & & \\ \vdots & \vdots & & \vdots & \ddots & & \\ * & * & \cdots & * & * & 1 & \\ * & * & \cdots & * & * & * & 1 \end{bmatrix}$$

Toeplitz Matrix:

$$\begin{bmatrix} 1 & & & & & & \\ \alpha_1 & 1 & & & & & \\ \alpha_2 & \alpha_1 & 1 & & & & \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 & & & \\ \vdots & \vdots & & \vdots & \ddots & & \\ \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_2 & \alpha_1 & 1 & \\ \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_3 & \alpha_2 & \alpha_1 & 1 \end{bmatrix}$$

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Dense Matrix:

- Must choose $\Theta\left(n^2\right)$ random matrix entries.

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- Easy $\Theta\left(n \log n \log\log n\right)$ matrix-vector products, good for sparse A.

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Toeplitz Matrix:

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- Easy $\Theta\left(n \log n \log\log n\right)$ matrix-vector products, good for sparse A.
- Existing research conditioning with Toeplitz matrices.

It is known that unit triangular Toeplitz matrices can be used to precondition for **generic rank profile** (all square leading principal submatrices are nonsingular).

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Theorem (Kaltofen and Saunders, 1991)

When post-multiplying a nonsingular matrix by a random unit lower triangular Toeplitz matrix, the product will (with high probability) have generic rank profile.

Recall: $A = USV$ with S in Smith form, U, V unimodular. We show that for the Hermite form of A to be in triangular Smith form, it suffices to ensure that the Smith multiplier V has generic rank profile.

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Our first main result is to show that Toeplitz matrices are sufficient preconditioners for triangular Smith form:

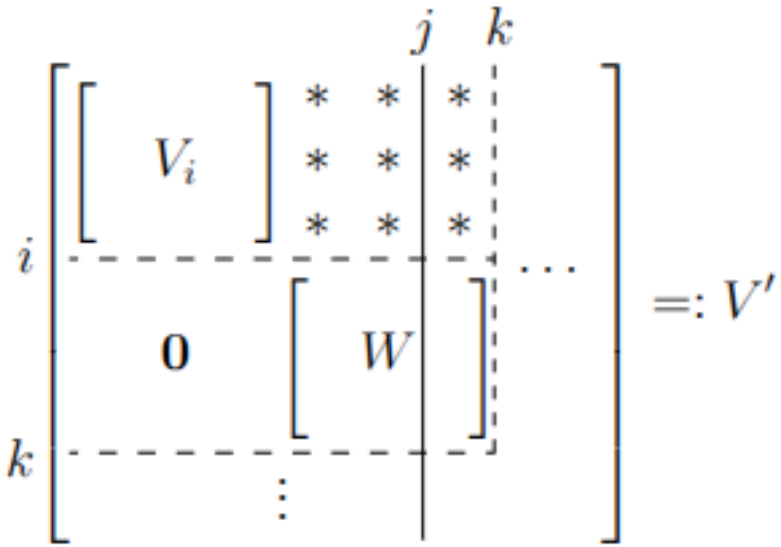
Theorem (Main Result #1)

When post-multiplying an input matrix by a random unit lower triangular **Toeplitz** *matrix, the resulting matrix will have the same Smith form, but have its Hermite form be in triangular Smith form with probability at least*

$$1 - \frac{dn^2(n+1)}{2|S|}.$$

The essence of the proof is as follows.

Firstly, we consider all irreducibles $p \in \mathbf{K}[x]$ that divide $\det A$. For arbitrary p, we find a condition on V that is equivalent to the Hermite form being in triangular Smith form with respect to p.



For this condition to be satisfied for all p, it suffices for V to have generic rank profile over $\mathbf{K}[x]/(p)$ for all p.

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Thankfully, the Smith and Hermite forms are **canonical forms**, and therefore do not change under field extensions.

The following definition is important for the statement of Main Result #2:

Definition

A nonsingular matrix $A \in \mathbb{R}^{n \times n}$ has *leading Smith determinants* if

$$\gcd(\det A_i, \det A) = s_i^*$$

for all $i = 1, \ldots, n$.

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Matrices with this property are used in, for example, the Smith form algorithm of Storjohann and Labahn, 1997.
Recall that $A = WH$ with H in Hermite form W unimodular. We show that, for a matrix to have leading Smith determinants, it suffices that:

- H is in triangular Smith form (see Result #1),
- W has generic rank profile.

Using our other main result, combined with the generic rank profile result by Kaltofen and Saunders, 1991, we can precondition for leading Smith determinants.

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Theorem (Main Result #2)

When pre- (and post-) multiplying an input matrix by a random unit upper (respectively lower) triangular **Toeplitz** *matrix, the resulting matrix will have leading Smith determinants with probability at least*

$$1 - \frac{dn^2(n+1)}{|S|}.$$

To conclude:

- Instead of a dense matrix, one can use structured matrix preconditioners to aid in Smith form computation.
- This can be especially useful when the input matrix is sparse.
- It may also be useful in other applications, such as calculating the *Frobenius normal form* of a sparse matrix, which can be done through calculating the Smith form.

Thank You!

References I

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