Frechet distance between polygonal curves

As a measure for the resemblance of curves in arbitrary dimensions we consider the so-called Fréchet distance, which is compatible with parametrizations of the curves. For polygonal chains $P$ and $Q$ consisting of $p$ and $q$ edges an algorithm of runtime $O(pq \log ( ...$

Figures here, notes follow.

![Diagram](image)

Fig. 3. Diagram for polygonal chains $P, Q$ and the given $\varepsilon$
Frechet distance between polygonal curves

![Diagram of intervals of free space on the boundary of a cell.]

**Efficient algorithms for geometric optimization**

PK Agarwal, M Sharir - ACM Computing Surveys (CSUR), 1998 - dl.acm.org

Abstract We review the recent progress in the design of efficient algorithms for various problems in geometric optimization. We present several techniques used to attack these problems, such as parametric searching, geometric alternatives to parametric searching, ...
Frechet distance between polygonal curves

Distances - to measure similarity
e.g. edit distance in strings (can be formulated as distance in a graph, "reconfiguration")

Distance between curves in the plane

similar

not

Hausdorff distance between $A$ and $B$.
- for each point $a \in A$ take min distance to $B$
- for each point $b \in B$ take min distance to $A$
take max of these (need sup, inf in general)
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Hausdorff distance does not capture similarity only takes into account the set of points, not their order on the curve

\[ \gamma : [0, 1] \rightarrow \mathbb{R}^2 \]

same curve may have different parameterizations

polygonal curve composed of n line segments
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Person moves on curve A \( \rightarrow \) never backwards
dog on curve B

Minimum length of leash: \( \text{Frechet distance} \)

More formally (Frechet 1906)

\[
\delta_F(\alpha, \beta) = \inf_{\text{reparameterizations } \alpha', \beta'} \max_{t \in [0,1]} \| \alpha'(t) - \beta'(t) \|
\]

How long the leash must be
Frechet distance between polygonal curves

Alt, Godau '95

- Algorithm to compute Frechet distance between 2 polygonal curves with $a$, $b$ vertices
  - Decision version $O(ab)$
  - Computing $O(ab \log ab)$

Main Idea

- Choosing position on $\alpha$, on $\beta$ gives a pt. in rectangle $d(\alpha(s), \beta(t))$
- Leash length for those positions
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For decision problem, block off pts in rectangle where leash is ≥ ε.

ε-diagram

See Fig. 3.

Lemma Frechet distance is ≤ ε iff there is a monotone path from (0,0) to (1,1) in the free space of the ε-diagram.

monotone = x coord never decreases

Question: what does the length of the path correspond to?
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\[ \varepsilon \)-diagram has \( a \times b \) rectangles - each corresponds to one segment of \( \alpha \) versus one segment of \( \beta \).

**Lemma.** Inside one rectangle the free space is an ellipse (Fig. 4).

Along each side, the free space is an interval.

If we know the subintervals on left & bottom reachable via monotone path from \((0, 0)\) then we can compute same for right/top in constant time.
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This solves decision problem in \( O(ab) \) time.

To actually compute the Frechet distance:

We need \( \epsilon > d(\alpha(0), \beta(0)) \) initial and final
\( \epsilon > d(\alpha(1), \beta(1)) \) leash lengths.

Starting from this \( \epsilon_0 \), imagine increasing \( \epsilon \).

Combinatorial changes at critical values of \( \epsilon \):

1. a new passage opens between neighboring cells
2. a new horizontal/vertical passage opens.

1 corresponds to \( \alpha \)
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Idea:
- determine all critical values (there are $O(a^2 b + b^2 a)$ of them)
- sort them
- do binary search for $\varepsilon$ (use decision alg.)

\[ \text{Time } O\left( (a^2 b + b^2 a) \log(ab) + ab \log(ab) \right) \]

\begin{itemize}
\item \underline{sort}
\item \underline{decision binary search}
\end{itemize}

\begin{itemize}
\item all $\varepsilon$ values
\item test this $\varepsilon$ and recurse on appropriate side
\end{itemize}
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For improved run-time of $O(ab \log(ab))$
use Megiddo's parametric search.

(Cole's variant using Ajtai-Komlós-Szemerédi sorting network — high constant and impractical).

See survey paper by Agarwal & Sharir.
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Further work

- can get run time below $O(n^2 \log n)$ - at least for $(1+\varepsilon)$-approx.

- Frechet distance inside a polygon - the leash must remain inside

- Frechet distance between 2 surfaces (up one dimension)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{frechet_distance.png}
\caption{Initial leash}
\end{figure}

E.g. the curves themselves might act as boundaries