Applications of Delaunay Triangulations

- All Nearest Neighbours

Nearest Neighbour Graph:

NN(\(P\)) - directed graph on vertex set \(P\)
with directed edge \((a, b)\) if \(a\)'s nearest neighbour is \(b\)

has out-degree 1
what is in-degree?

\[ \text{max in-degree is 6} \]
(with ties)

Claim \(\text{NN}(P) \subseteq \text{D}(P)\)

if \(p_j\) is nearest neighbour of \(p_i\) then \((p_i, p_j) \in \text{D}(P)\)

To find \(\text{NN}(P)\), find \(\text{D}(P)\) and test each the neighbours of each vertex to find the closest one.

\(O(n)\) after finding \(\text{D}(P)\).

Similarly, can find the \(k\) nearest neighbours of each point using the \(k\)th order Voronoi diagram (later)

- see slide
Application 2 - Min. Spanning Tree of points in the plane using Euclidean edge lengths:

- standard MST alg. is at least $\Omega(m)$
  and we have $m = \Theta(n^2)$

**Lemma** The MST is a subgraph of the Delaunay Triang.
Thus, can run standard MST alg. on Delaunay triang
- $O(n \log n)$ time alg.

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**Proof of Lemma**

Removing edge $(a, b)$ from MST separates tree into $T_a, T_b$

**Claim.** There is no point in circle $C$ with diameter $(a, b)$.
Then $e \in D(p)$ by empty circle property.

**Pf. of Claim** Suppose point $p$ is in $C$.

$|ap| < |ab|$ so if $p \in T_b$ use $ap$ instead of $ab$

$|bp| < |ab|$ so if $p \in T_a$ use $bp$ contra to having a MST.
Application 3. Relative Neighbourhood & Gabriel Graphs

"proximity graphs"

\[ \text{RNG - relative neighbourhood graph} \]

Connect \( ab \) if this \( \text{lune} \) is empty.

i.e., \( \exists p \) closer to both \( a \) and \( b \) than \( |ab| \)

\[ \text{GG - Gabriel graph} \]

Edge \( (a, b) \) if \( a \) and \( b \) circle with diam. \( ab \) is empty.

\[ \text{NN}(P) \leq \text{MST}(P) \leq \text{RNG}(P) \leq \text{GG}(P) \leq \mathcal{D}(P) \]

The Gabriel graph and the Relative Neighbourhood graph can be computed in time \( O(n) \) given the Delaunay graph. (This is not obvious as it was for Nearest Neighbourhood Graph.)
Application 4: finding largest empty circle

Given $n$ points in a convex boundary polygon $B$, find the largest empty circle with center in $B$.

E.g.- locate a new store location among existing stores
- locate a nuclear waste dump among cities.

Lemma The center of the largest empty circle is either
- a Voronoi vertex
- the intersection of a Voronoi edge with an edge of $B$
- a vertex of $B$

Proof idea: any other circle, can be enlarged
- if $x$ interior to $B$ and interior to Vor. region
  - if $C$ touches no pt, increase $r$
  - else keep contact with pt, move x, incr. $r$
- if $x$ on Voronoi edge, move $x$ along edge to increase $r$
- only stopped when $x$ is Vor. vertex, or Vor. edge intersect edge of $B$
- or vertex of $B$
to solve largest empty circle problem:
compute Voronoi diagram \( O(n \log n) \)

then test all above possible points
\( O(n) \) Voronoi vertices
\( O(n) \) Voronoi edges, each intersects \( B \) at most once
vertices of \( B \).

connections between Del. triang. and convex hull
Given \( p_1, \ldots, p_n \in \mathbb{R}^2 \) project them up onto parabola \( z = x^2 + y^2 \)
\( p = (x_p, y_p) \rightarrow \hat{p} = (x_p, y_p, x_p^2 + y_p^2) \)

Theorem. The lower convex hull of \( \hat{p}_1, \ldots, \hat{p}_n \),
projected back to the plane, is the Delaunay triangulation of \( p_1, \ldots, p_n \).
Claim 1: Points in $\mathbb{R}^2$ are co-circular
iff their projections are co-planar.

pf: eqn of circle center $(a,b)$ radius $r$
\[(x-a)^2 + (y-b)^2 = r^2\]
\[(x^2+y^2) - 2ax - 2by + (a^2+b^2-r^2) = 0\]
\[\Rightarrow \text{this is the eqn of a plane in } x, y, z = x^2+y^2.\]

Claim 2: pts outside circle map to points above plane
"inside" "below".

pf of thm: lower
\[\hat{p}_i \hat{p}_j \hat{p}_k \text{ form a face of } CH(\hat{P})\]
iff there is a plane through $\hat{p}_i \hat{p}_j \hat{p}_k$ with all
other points above the plane
iff there is a circle through $p_i, p_j, p_k$ with
all other points outside
iff $p_i, p_j, p_k$ form a triangle in $D(P)$.

Note: the tangent planes of points $\hat{p}$ on
the parabola give the Voronoi diagram.
the upper envelope of $D(P)$ in 3D is a 4D(lower) CH.

Note: this also true up one dimension; $D(p)$ in 3D is a 4D(lower) CH.
Algorithms for Del. triang. / Voronoi diagram
Ex. We can get either one from the other in $O(n)$ time
- 3D CG alg can compute 2D Del. triang $O(n \log n)$
  - divide and conquer.
- or can do divide & conquer directly
  - the merge step is complicated.
- Fortune - sweep line alg. 1987 - today
- randomized incremental alg. - next week

Fortune's sweep line alg.
- sweep horizontal line across plane top to bottom.

Tikky issue
we encounter $V(p)$
before we see $p$
Solution

Find the Voronoi diagram of points plus halfplane $y \leq k$ from $k = \infty$ to $k = -\infty$.

For $k = \infty$ every point has distance 0 to the half-plane.

For $k = -\infty$ every point is closest to some site—so we have the Voronoi diagram.

In between for 1 site ($n=1$)

- See slides.
Variants of Voronoi diagrams
- sites more general than points
- in 3D or kD
- farthest point Voronoi diagram
- weighted Voronoi diagram
- other metrics instead of Euclidean

See slides