Voronoi diagram

Given points \( P = \{ p_1, \ldots, p_n \} \) in the plane
the Voronoi region of \( p_i \) is \( V(p_i) \)

\[ V(p_i) = \{ x \in \mathbb{R}^2 : d(x, p_i) \leq d(x, p_j) \quad \forall j \neq i \} \]

\( p_i \) is called a site

The Voronoi diagram \( V(P) = \) all Voronoi regions.

To see that region boundary consists of straight segments

\[ H(p_i, p_j) = \] points closer to \( p_i \) than \( p_j \)
\[ = \text{half-plane} \]

\[ V(p_i) = \bigcap_{j \neq i} H(p_i, p_j) \]

So Voronoi diagram is a planar graph with

vertices, (straight) edges, faces = Voronoi regions

Properties of Voronoi diagrams

we will assume \( n \leq 4 \) sites on a circle
- Then Voronoi vertices have degree 3.
- \( V(p_i) \) is convex (as already shown, it’s the intersection of half-planes)
- for Voronoi vertex \( v \) with closest sites \( p_i, p_j, p_k \), the circle centered at \( v \) through \( p_i, p_j, p_k \) contains no other site
- \( V(p_i) \) is unbounded iff \( p_i \) is on the convex hull

\[ \iff \text{this line is in } V(p_i) \]

\[ \Rightarrow \text{some infinite ray is in } V(p_i) \]

as \( x_i \to \infty \)

get empty half-plane thru \( p_i \)
Voronoi diagram has $O(n)$ vertices and $O(n)$ edges.

Buler's formula $\nabla - e + f = 2$

$f = n$ # sites

Note: we deal with inf. edges by adding "pt at infinity" where all unbounded edges meet

every vertex has deg. $\geq 3$

$2e = \Sigma \text{degrees} \geq 3\nu$

so $\nu \leq \frac{2}{3} e$

$2 = \nabla - e + f \leq \frac{2}{3} e - e + n = -\frac{1}{3} e + n$

so $e \leq 3(n-2)$ and $\nu \leq 2(n-2)$

Note: this strange case happens only for collinear pts

Delaunay triangulation

- a combinatorial structure that captures the Voronoi diagram.

Delaunay triangulation $D(P)$

of points $P = \{P_1, \ldots, P_n\}$

is graph

vertices: $P_1, \ldots, P_n$

edge $(P_i, P_j)$ iff $V(P_i)$ and $V(P_j)$ share an edge.

This is the planar dual of $V(P)$.

see slides.

Ex. Find an example where a Delaunay edge does not cross its dual edge.
Properties of Delaunay triangulations

- faces are triangles

because Voronoi vertices have degree 3

\[(p_i, p_j) \text{ is an edge of } \mathcal{D}(P) \text{ if and only if there is an empty circle through } p_i p_j\]

\[\Rightarrow \] \(V(p_i) \) and \(V(p_j)\) share an edge.

Any point on that edge is the center of an empty circle through \(p_i\) and \(p_j\).

\[\Leftarrow \] The center of an empty circle through \(p_i\) and \(p_j\) is equally close to \(p_i\), \(p_j\), and not closer to any other site.

Thus, it's on the Voronoi edge between \(V(p_i)\) and \(V(p_j)\). So \((p_i, p_j) \in \mathcal{D}(P)\)
Properties of Delaunay triangulations
- no two edges of \( D(P) \) cross.

Note that this must be proved

\[
\text{not true for general planar maps}
\]

Pf. Suppose \((u, v)\) an edge of \( D(P) \)

then \( \exists \) empty circle thru \( u, v \).

Can we have an edge \((x, y)\) crossing \((u, v)\)?

Need empty circle thru \( x, y \).

Expand \( C \) to touch \( u, v \), contain \( x, y \). Not possible.

\[
\text{boundary of } \ Q(P) \text{ is convex hull of sites}
\]

\[
\begin{align*}
&\iff e = (u, v) \text{ an edge of } C^T \\
&\text{then for } x \text{ on perp. bisector of } e, \ x \to \infty, \\
&\text{ } x \text{ is in } V(u) \cap V(v) \\
\Rightarrow &\text{if } e = (u, v) \text{ boundary of } D(P) \text{ then} \\
&\text{empty circle thru } u, v \text{ can expand to } \infty \text{ (else get } \Delta) \\
&\text{so } (u, v) \text{ an edge of } C^T.
\end{align*}
\]
triangular

- \( P_i, P_j, P_k \) is a \( \triangledown \) face of \( D(P) \) iff there is an empty circle through \( P_i, P_j, P_k \).

\[ \iff \] by previous result we have edges \( P_i, P_j, P_k \), \( P_i P_k \), \( P_k P_i \), and nothing inside \( \triangledown \) a face.

\[ \Rightarrow \]

Show that circle thru \( P_i, P_j, P_k \) is empty.

No points inside \( \Delta \) (it's a face).

Can there be a point here?

No - there would be no empty circle thru \( P_j, P_k \).

Alternative defn of Delaunay triangulation
- for each empty circle through 3 points, add a triangle.

See slide.