RECALL

Plan for the next two lectures

- definitions of convex hull, and more about convex polyhedra
- divide and conquer for convex hull in 3D
- randomized algorithm for convex hull in any dimension
- also discussion of convex hull in higher dimensions
Equivalent definitions of Convex Hull of a set of points S

1. intersection of all convex sets containing S

2. all convex combinations of points in S

3. intersection of all half-spaces containing S

4. in 2D. A convex polygon P whose vertices are a subset of S, and such that all points of S are inside P

5. A convex polyhedron P whose vertices are a subset of S and such that all points of S are inside P

A convex set is such that if points p, q are in the set then line segment pq is in the set.

A convex combination of points p1, p2, ..., pn is ...

\[ \sum_{i=1}^{n} \lambda_i p_i \quad \text{for} \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0 \]
Equivalent definitions of Convex Hull of a set of points S

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A convex polyhedron (in dimension $d$) is

\[ \exists x : \mathbb{R}^d \ni Ax \leq b \]

i.e. a bounded intersection of half-spaces

and we are assuming the set is bounded.

caution: the term “polyhedron” means different things in different areas (convex/non-convex, bounded/unbounded)
A cube in 3D is the intersection of 6 half-spaces \( \mathbb{R}^3 \) of the form \( x_i \leq 1 \) and \( 0 \leq x_i \leq 1 \).

A face is any \( \exists x \in \mathbb{R}^d : Ax = b \) and some inequalities are changed to equalities.

For example, the front face of the cube has \( x_2 = 1 \).
The face lattice of a convex polyhedron

- 8 cubes
- 24 squares
- 32 edges
- 16 vertices
Size of convex hull of n points in d-dimensions

Recall from last day: in 3D the number of faces (facets) and size of face lattice are $O(n)$ by Euler’s formula.

**McMullen’s Upper bound Theorem**

For a convex polyhedron in $d$ dimensions (d fixed) with $n$ vertices the worst case number of faces is

$$\Theta \left( n \left\lfloor \frac{d}{2} \right\rfloor \right)$$

The number of facets has the same bound (we get a $2^d$ constant appearing).

In fact, McMullen gave more exact bounds — the above asymptotic bound is easier to show [https://graphics.stanford.edu/courses/cs268-11-spring/notes/upper_bound_theorem.pdf](https://graphics.stanford.edu/courses/cs268-11-spring/notes/upper_bound_theorem.pdf)

For $d=2,3$ bound is $\Theta(n)$.
For $d=4,5$ bound is $\Theta(n^2)$

d=4 matters! One application of 4D convex hull is to find 3D Delaunay triangulations.
The bound of $\Theta \left( n^{\frac{d}{2}} \right)$ is realized by a cyclic polytope — the convex hull of $n$ points on the moment curve.

Moment curve = $\left\{ (t, t^2, \ldots, t^d) : t \in \mathbb{R}^2, t_1 \leq t_2 \leq \ldots \leq t_n \right\}$

Place $n$ points on the moment curve.

Claim. The number of facets of their convex hull is $\Theta \left( n^{\frac{d}{2}} \right)$.
**[Randomized] Incremental Convex Hull Algorithm**

We will describe the algorithm for 3D though it does extend to general dimensions.

Assume no 4 points lie on a plane (this means that all faces will be triangles). See [CG] book for details on more general case.

Idea: Add the points one by one in random order.

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**Diagram Description:**

- **Horizon edge** connects points in the hull.
- **New face** is added as a result of adding a point.

*Computational Geometry, de Berg, et al.*
Details of Randomized Incremental Convex Hull Algorithm
Randomized Incremental Convex Hull Algorithm

- expected run time $O(n \log n)$ in 3D
- expected run time for $d \geq 4$ is $O\left(n^{\left\lfloor d/2 \right\rfloor}\right)$

and we can use linear programming instead of the conflict graph

Recall: size of convex hull (facets or whole face lattice) is $\Theta\left(n^{\left\lfloor d/2 \right\rfloor}\right)$

Combining lower bound for $d=2$ and lower bound due to output size, we get lower bound of

$$\sum_2 (n \log n + n^{\left\lfloor d/2 \right\rfloor}) \quad (d \text{ constant})$$

So randomized incremental algorithm is optimal in the worst case.

Is there a deterministic (non-randomized) algorithm?

Yes. Chazelle ’93 by derandomizing the above algorithm (choose an order of points that has the good properties of a random order).

Output sensitive? Lower bound is $\Omega(h + n \log h)$.
Chan got $O(n \log h)$ for $d=2,3$. Seidel ’86 achieved $O(n^2 + h \log h)$ for fixed $d$. 
RECALL

The convex hull problem

Given a set of $n$ points in $d$-dimensions, find their convex hull (as an intersection of half-spaces)

or sometimes we ask for the whole face lattice

Dual problem:

Given a set of $m$ halfspaces in $d$-dimensions, find their intersection (as the set of vertices of the polyhedron)

or sometimes we ask for the whole face lattice

In fact, these two problems are the same by a duality map.
Duality between points and hyperplanes