RECALL

Plan for the next two lectures

- definitions of convex hull, and more about convex polyhedra
- divide and conquer for convex hull in 3D
- randomized algorithm for convex hull in any dimension
- also discussion of convex hull in higher dimensions

RECALL

Equivalent definitions of Convex Hull of a set of points S

1. intersection of all *convex sets* containing S

2. all *convex combinations* of points in S

3. intersection of all half-spaces containing S

4. in 2D. A *convex polygon* P whose vertices are a subset of S, and such that all points of S are inside P

5. A *convex polyhedron* P whose vertices are a subset of S and such that all points of S are inside P

A *convex set* is such that if points p, q are in the set then line segment pq is in the set.

A *convex combination* of points p1, p2, . . . , pn is . . .

$$\sum_{i=1}^{n} \lambda_i p_i \quad \text{for} \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0$$

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A *convex polyhedron* (in dimension *d*) is

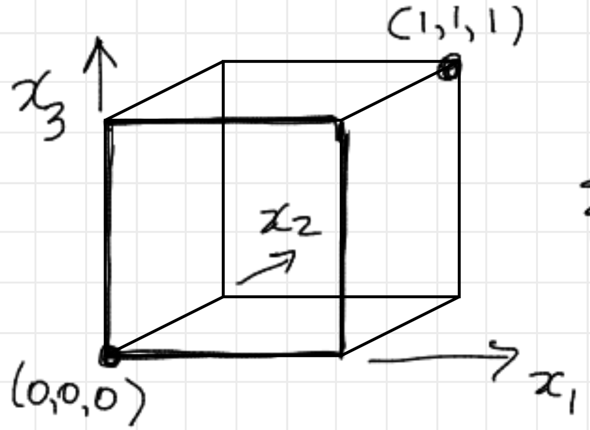
a bounded intersection of half-spaces

i.e. $\{x: \mathbb{R}^d : Ax \le b\}$ A – $m \times d$ matrix
b – m-vector

and we are assuming the set is bounded.

caution: the term “polyhedron” means different things in different areas (convex/non-convex, bounded/unbounded)

Face of a polyhedron




A cube in 3D is the intersection of 6 half-spaces

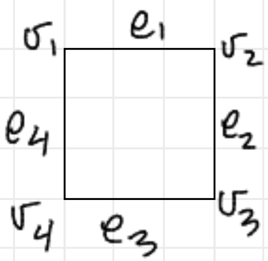
$$\{(x_1, x_2, x_3) : 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1\}$$

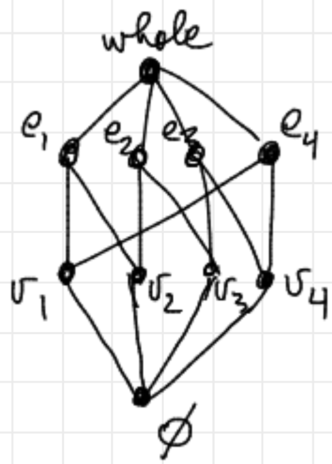
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

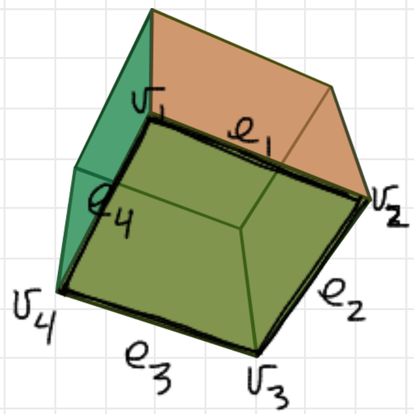
A <u>face</u> is $\{x \in \mathbb{R}^d : Ax \le b\}$ and some inequalities are changed to equalities.

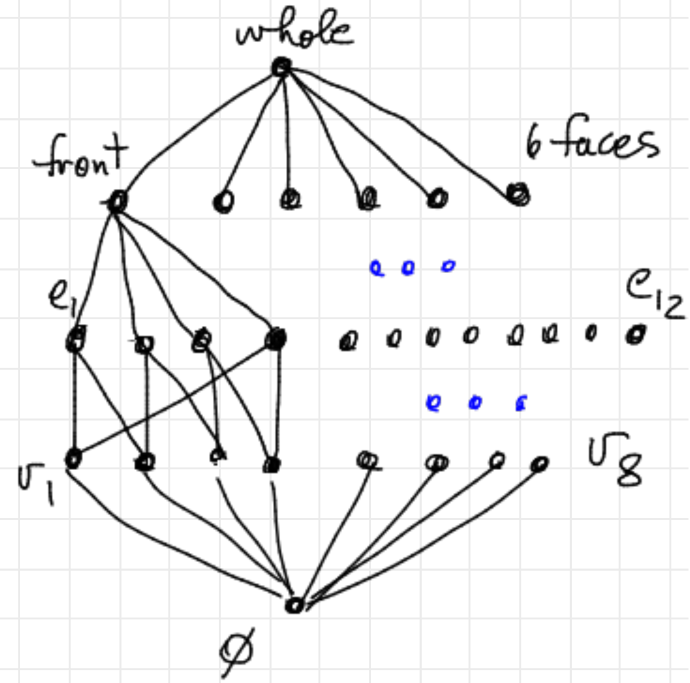
e.g. front face of cube has $x_2 = 1$.

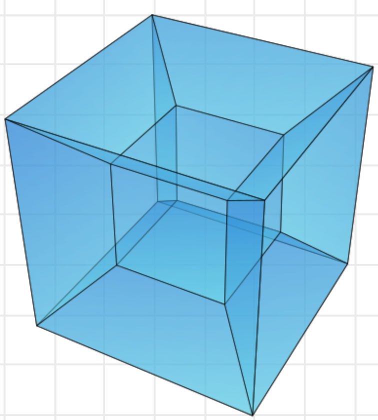
The face lattice of a convex polyhedron



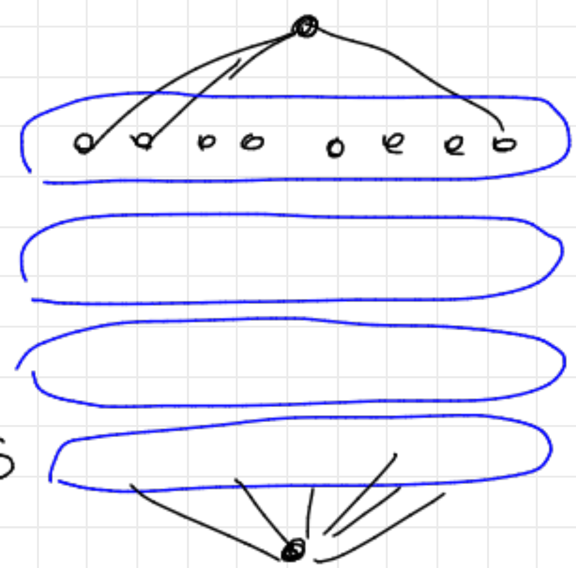



8 cubes

24 squares

32 edges

16 vertices



**Size of convex hull of n points in d-dimensions**

Recall from last day: in 3D the number of faces (facets) and size of face lattice are O(n) by Euler's formula.

**McMullen's Upper bound Theorem**
For a convex polyhedron in *d* dimensions (d fixed) with *n* vertices the worst case number of faces is

$$\Theta\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$$

The number of facets has the same bound (we get a 2^d constant appearing).

In fact, McMullen gave more exact bounds — the above asymptotic bound is easier to show https://graphics.stanford.edu/courses/cs268-11-spring/notes/upper_bound_theorem.pdf

For d=2,3 bound is Theta(n).
For d=4,5 bound is Theta($n^2$)

d=4 matters! One application of 4D convex hull is to find 3D Delaunay triangulations.

The bound of $\Theta(n^{\lfloor d/2 \rfloor})$

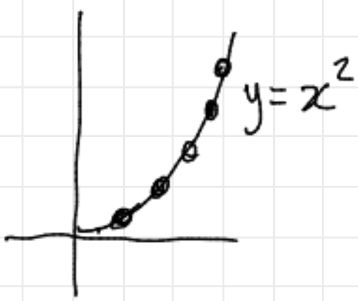
is realized by a *cyclic polytope* — the convex hull of n points on the *moment curve*

moment curve = $\{ (t, t^2, \ldots, t^d) : t \in \mathbb{R} \}$

$t_1 \leq t_2 \leq \cdots \leq t_n$

Place n points on the moment curve.
Claim. The number of facets of their convex hull is $\Theta(n^{\lfloor d/2 \rfloor})$

in 2D




in 4D

can prove:

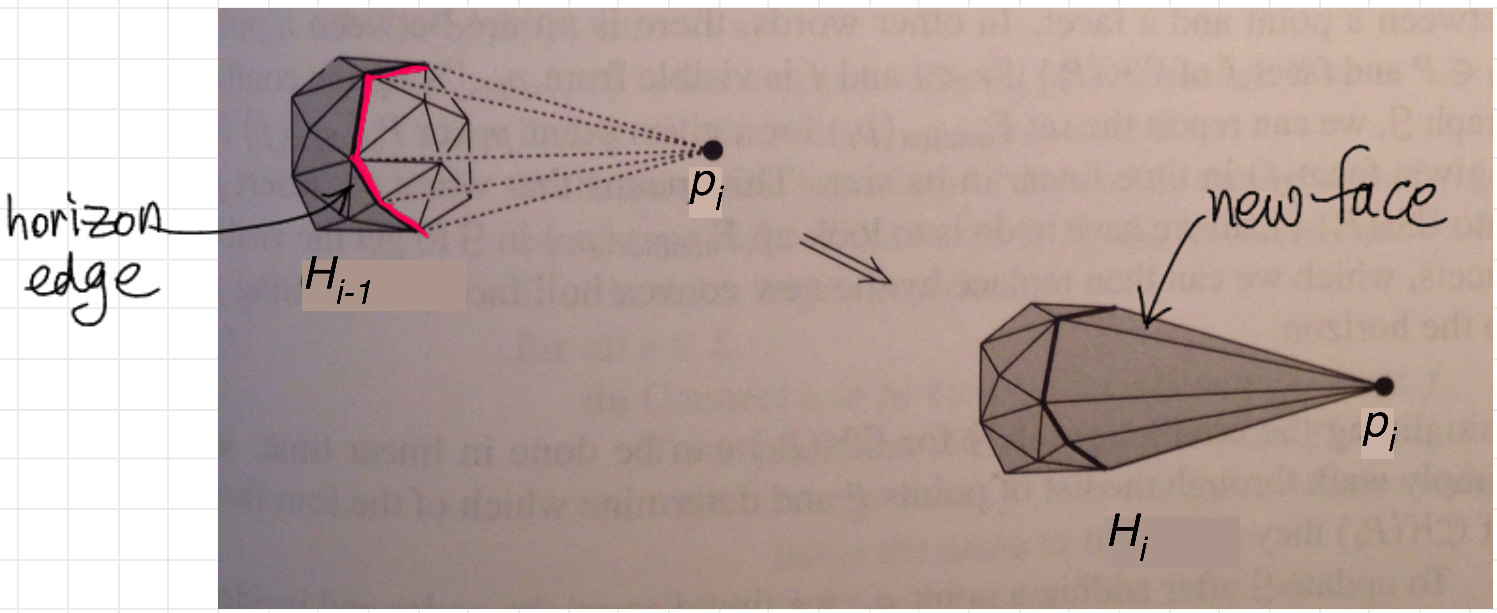
- Every pair $t_i t_j$ gives an edge of Convex Hull $- \Theta(n^2)$
- Every 4-tuple $t_i \, t_{i+1} \, t_j \, t_{j+1}$ gives a facet of Convex Hull
  - there are $\Theta(n^2)$ facets.

## [Randomized] Incremental Convex Hull Algorithm

We will describe the algorithm for 3D though it does extend to general dimensions.

Assume no 4 points lie on a plane (this means that all faces will be triangles).
See [CG] book for details on more general case.

Idea: Add the points one by one in random order.




Computational Geometry, de Berg, et al.

Details of Randomized Incremental Convex Hull Algorithm

**Randomized Incremental Convex Hull Algorithm**

- expected run time O(n log n) in 3D
- expected run time for d >= 4 is $O(n^{\lfloor d/2 \rfloor})$

  and we can use linear programming instead of the conflict graph

Recall: size of convex hull (facets or whole face lattice) is $\Theta(n^{\lfloor d/2 \rfloor})$

Combining lower bound for d=2 and lower bound due to output size, we get lower bound of $\Omega(n \log n + n^{\lfloor d/2 \rfloor})$ (d constant)

So randomized incremental algorithm is optimal in the worst case.

Is there a deterministic (non-randomized) algorithm?

Yes. Chazelle '93 by derandomizing the above algorithm (choose an order of points that has the good properties of a random order).

Output sensitive? Lower bound is Omega(h + n log h).
Chan got O(n log h) for d=2,3. Seidel '86 achieved O(n^2 + h log h) for fixed d.

RECALL

The convex hull problem

Given a set of n points in d-dimensions, find their convex hull
(as an intersection of half-spaces)

or sometimes we ask for the whole face lattice

Dual problem:

Given a set of m halfspaces in d-dimensions, find their intersection
(as the set of vertices of the polyhedron)

or sometimes we ask for the whole face lattice

In fact, these two problems are the same by a *duality* map.

## Duality between points and hyperplanes