Recall:

- convex hull in 2D
- definitions of convex hull, convex polyhedra
- divide and conquer for convex hull in 3D
A cube in 3D is the intersection of 6 half-spaces

\[ \bar{c}(x_1, x_2, x_3): 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1/3 \]

The face is \( \{ x \in \mathbb{R}^d : Ax \leq b \} \) and some inequalities are changed to equalities.

e.g. front face of cube has \( x_2 = 1 \).
The face lattice of a polyhedron

2D

4D

3D

A point is \((x_1, x_2, x_3, x_4)\)

8 cubes
24 squares
32 edges
16 vertices

whole
front

6 faces
COMPUTING CONVEX HULLS IN 3-SPACE

facets of \( \text{CH}(P_{r-1}) \)—the ones on the front side—but others will be invisible because they are on the back side. The visible facets form a connected region on the surface of \( \text{CH}(P_{r-1}) \), called the visible region of \( p_r \) on \( \text{CH}(P_{r-1}) \), which is enclosed by a closed curve consisting of edges of \( \text{CH}(P_{r-1}) \). We call this curve the horizon of \( p_r \) on \( \text{CH}(P_{r-1}) \). As you can see in Figure 11.2, the projection of the horizon is the boundary of the convex polygon obtained by projecting \( \text{CH}(P_{r-1}) \) onto a plane, with \( p_r \) as the center of projection.

What exactly does “visible” mean geometrically? Consider the plane containing a facet \( f \) of \( \text{CH}(P_{r-1}) \). By convexity, \( \text{CH}(P_{r-1}) \) is completely contained in one of the closed half-spaces defined by \( h_f \). The face \( f \) is visible from a point if that point lies in the open half-space on the other side of \( h_f \).

The horizon of \( p_r \) plays a crucial role when we want to transform \( \text{CH}(P_{r-1}) \) to \( \text{CH}(P_r) \): it forms the border between the part of the boundary that can be kept—the invisible facets—and the part of the boundary that must be replaced—the visible facets. The visible facets must be replaced by facets connecting \( p_r \) to its horizon.

Before we go into more details, we should decide how we are going to represent the convex hull of points in space. As we observed before, the boundary of a 3-dimensional convex polytope can be interpreted as a planar graph. Therefore we store the convex hull in the form of a doubly-connected edge list, a data structure developed in Chapter 2 for storing planar subdivisions. The only difference is that vertices will now be 3-dimensional points. We will keep the convention that the half-edges are directed such that the ones bounding any face form a counterclockwise cycle when seen from the outside of the polytope.

Incremental Convex Hull

Back to the addition of \( p_r \) to the convex hull. We have a doubly-connected edge list representing \( \text{CH}(P_{r-1}) \), which we have to transform into a doubly-connected edge list for \( \text{CH}(P_r) \). Suppose that we knew all facets of \( \text{CH}(P_{r-1}) \) visible from \( p_r \). Then it would be easy to remove all the information stored for these facets from the doubly-connected edge list, compute the new facets connecting \( p_r \) to the horizon, and store the information for the new facets in the doubly-connected edge list. All this will take linear time in the total complexity of the facets that disappear.

There is one subtlety we should take care of after the addition of the new point.
The convex hull problem
Given $n$ points in $\mathbb{R}^d$ find their convex hull
- as an intersection of half-spaces
- or as the face lattice

Dual problem
Given $m$ half-spaces in $\mathbb{R}^d$ find their intersection
- as the vertices of the polyhedron
- or as the face lattice

In fact, these 2 problems are the same
via a duality map (more later)
References


asymptotic version of McMullen’s Upper Bound theorem: