Given points in $d$-dimensional space, find a good “container” = convex polytope. Many applications, e.g. collision detection, pattern recognition, motion planning . . .

In 2D, imagine putting a rubber band around the points

In 3D, wrap with shrink-wrap

More formally:

A set is **convex** if for every two points $p, q$ in the set, all points on the line segment $pq$ are also in the set.

The **convex hull** of set $S$ is the intersection of all convex sets that contain $S$.

Note that the convex hull of $S$ is convex. The fact that the convex hull of a set of points $S$ is a convex polytope whose vertices are points of $S$ requires a proof, which we will do later.
Plan for the next two lectures

- definitions of convex hull, and more about convex polyhedra
- divide and conquer for convex hull in 3D
- randomized algorithm for convex hull in any dimension
Equivalent definitions of Convex Hull of a set of points S

1. intersection of all *convex sets* containing S

2. all *convex combinations* of points in S

3. intersection of all half-spaces containing S

4. in 2D. A *convex polygon* \( P \) whose vertices are a subset of S, and such that all points of S are inside \( P \)

5. A *convex polyhedron* \( P \) whose vertices are a subset of S and such that all points of S are inside \( P \)

A *convex set* is such that if points \( p, q \) are in the set then line segment \( pq \) is in the set.

A *convex combination* of points \( p_1, p_2, \ldots, p_n \) is . . .
Equivalent definitions of Convex Hull of a set of points $S$

1. intersection of all *convex sets* containing $S$

2. all *convex combinations* of points in $S$

3. intersection of all half-spaces containing $S$

4. in 2D. A *convex polygon* $P$ whose vertices are a subset of $S$, and such that all points of $S$ are inside $P$

Illustration for 2D
Equivalent definitions of Convex Hull of a set of points $S$

1. intersection of all *convex sets* containing $S$

2. all *convex combinations* of points in $S$

3. intersection of all half-spaces containing $S$

How are these related?
Equivalence of 2 and 3 is proved using some version of Farkas’s Lemma

either $p$ is a convex combination of points of $S$
OR
there is a plane separating $p$ from $S$
AND NOT BOTH
Equivalent definitions of Convex Hull of a set of points S

1. intersection of all \textit{convex sets} containing S
2. all \textit{convex combinations} of points in S
3. intersection of all half-spaces containing S
4. in 2D. A \textit{convex polygon} P whose vertices are a subset of S, and such that all points of S are inside P
5. A \textit{convex polyhedron} P whose vertices are a subset of S and such that all points of S are inside P

A \textit{convex polyhedron} (in dimension \(d\)) is

\textit{caution: the term “polyhedron” means different things in different areas (convex/non-convex, bounded/unbounded)
Face of a polyhedron

A cube in 3D is the intersection of 6 halfspaces

\[ \{ (x_1, x_2, x_3) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1 \} \]

\[
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\leq
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

A face is \( \{ x \in \mathbb{R}^d : A x \leq b \} \) and some inequalities are changed to equalities.

E.g., front face of cube has \( x_2 = 1 \).
The face lattice of a convex polyhedron

- cubes
- squares
- edges
- vertices
The face lattice of a convex polyhedron

\[ \text{i-face} = \text{i dimensional face} \]
\[ \text{0-face} = \text{vertex} \]
\[ \text{1-face} = \text{edge} \]
\[ \text{(d-1)-face} = \text{facet} \]
\[ \text{d-face} = \text{the whole polyhedron} \]
Equivalent definitions of Convex Hull of a set of points $S$

1. intersection of all *convex sets* containing $S$

2. all *convex combinations* of points in $S$

3. intersection of all half-spaces containing $S$

4. in 2D. A *convex polygon* $P$ whose vertices are a subset of $S$, and such that all points of $S$ are inside $P$

5. A *convex polyhedron* $P$ whose vertices are a subset of $S$ and such that all points of $S$ are inside $P$

Equivalence of these definitions proved using:

Theorem [Minkowski, Weyl]  The set of all convex combinations of $p_1, \ldots, p_n$ is a bounded convex polyhedron whose vertices are a subset of $p_1, \ldots, p_n$
The convex hull problem

Given a set of \( n \) points in \( d \)-dimensions, find their convex hull (as an intersection of half-spaces)

or sometimes we ask for the whole face lattice

Dual problem:

Given a set of \( m \) halfspaces in \( d \)-dimensions, find their intersection (as the set of vertices of the polyhedron)

or sometimes we ask for the whole face lattice

In fact, these two problems are the same by a duality map (more later)
3D convex hull. Given $n$ points in 3D, find their convex hull, i.e. find the vertices, edges and faces of the convex hull.

What is the size of the convex hull?
3D Convex Hull Algorithms

Some of the 2D algorithms extend to 3D.

Exercise: Does the incremental algorithm extend? Is it $O(n \log n)$?

Divide and Conquer.

Basically the ONLY known $O(n \log n)$ 3D convex hull algorithm. Preparata and Hong 1977.

- sort points by x coordinate
- divide by orthogonal plane at median x coordinate into two sets of size $n/2$
- recurse on each side to find convex hulls $A$ and $B$
- combine $A$ and $B$ into one convex hull

If we combine in $O(n)$ we get $T(n) = 2T(n/2) + O(n)$ which yields $T(n) = O(n \log n)$

O'Rourke, Comp. Geom. in C
How to combine two disjoint convex hulls in $O(n)$

we must find the “band” of faces that cross our dividing plane and then discard “hidden” faces

1. find an edge $ab$ of the convex hull, $a$ in $A$, $b$ in $B$ and a plane through $ab$ with $A$ and $B$ to one side.
2. pivot the plane through $ab$ to find a face of the convex hull band
3. repeat until we wrap back to $ab$
4. remove hidden faces

O’Rourke, Comp. Geom. in C
1. find an edge ab of the convex hull, a in A, b in B and a plane through ab with A and B to one side.

2. pivot the plane through ab to find a face of the convex hull band

3. repeat until we wrap back to ab

4. remove hidden faces

**Details and Timing**

**Step 1.** Project to 2D and find lower bridge

**Step 2.**

Lemma. The next point, c, is a neighbour of a or b.

So, find “best” neighbour of a and “best” neighbour of b.

Lemma. If the next point, c, is a neighbour of b, then the next “best” neighbour of a remains the same.

Total time: Sum of all vertex degrees. This is O(n) because we have a planar graph.
Removing hidden faces can be done in $O(n)$ too.

Note that the cycle of “horizon” edges need not be simple.

This "horizon" is a simple cycle.

Here the horizon is not a simple cycle.

O’Rourke and Devadoss

O’Rourke, Comp. Geom. in C
Conventional wisdom was that the divide and conquer algorithm is hard to implement
e.g. see O’Rourke’s book, “Computational Geometry in C”, 1998.

However, there are now good implementations

https://cs.uwaterloo.ca/~tmchan/ch3d/ch3d.pdf
The Gift-wrapping algorithm extends to 3D, $O(nh)$, $h = \#$ faces of the convex hull.

Use the same kind of “wrapping” we just saw for divide and conquer.

Timothy Chan’s $O(n \log h)$ algorithm extends to 3D.

Recall it needs an $O(n \log n)$ algorithm (the divide and conquer algorithm) plus an $O(nh)$ algorithm (the gift-wrapping algorithm).

The step of finding the “extreme” point in each of the smaller convex hulls needs more detail.