Definitions

\[ Z_A(l) = \text{faces of } A \text{ cut by } l \]

\[ z_A(l) = \sum_{f \in Z_A(l)} \# \text{edges in } f \]

Note: This counts duplicates

\[ z_n = \max_{l \in A} z_A(l) \text{ over all possible } l, A \text{ of } n \text{ lines} \]

Zone of \( l_i \) has 6 faces

\[ z(l_i) = 3 + 3 + 4 + 3 + 2 + 3 = 18 \]

Ignoring rectangular boundary

Zone Theorem: \( z_n \) is \( O(n) \). (For nondegenerate case, \( z_n \leq 6n \).)

[Chazelle, Guibas, Lee '85
Edelsbrunner, O'Rourke, Seidel '86
Correct pf. for dim \geq 3 Edelsbrunner, Seidel, Sharir '93]

Consequence: the incremental alg. takes time \( O(n^2) \)
Proof

We will bound \( z_A(l) \) where \( A \) is an arrangement of \( n \) lines.

Rotate so \( l \) is horizontal.

Perturb slightly so no other line is horizontal (this will only increase zone complexity).

Any face in \( Z_A(l) \) has left and right boundary edges.

\[
Z_A(l) = \sum_{l \in A} z_l^L(l) + \sum_{l \in A} z_l^R(l)
\]

Claim \( z_l^L(l) \leq 5n \) \([\leq 3n \text{ for non-degenerate}]\).

Then \( Z_n \) is \( O(n) \).

Pf of claim. By induction on \( n \).

Base: \( n = 1 \). Then zone of \( l \) has 2 faces, 1 left edge.

General case

Find rightmost intersection \( x \) of \( l \) with line \( l \) — say line \( l_t \).

Find first intersection on \( l_t \) above/below \( l \).
Note: $l_t$ has + slope, $l_b$ has - slope because $x$ was rightmost intersection.

New left edges:
- portion of $l_i$
- left edge on $l_t$ splits in two
- left edge on $l_b$ splits in two

Observe: no part of $l_i$ above $l_t$ is a left edge (ditto below)
no other existing left edge cut by $l_i$

change in # left edges: $+3$
this gives total of $3n$ left edges
but this ignored possible degeneracy

what if another line $l_j$ goes though $x$?
- still split $l_t$ and $l_b$
- but $l_i$ gets 2 left edges
- and $l_j$ gets split

change in # left edges: $+5$
by induction # left edges $\leq 5(n-1) + 5 = 5n$.

*and note that increase is less if more lines go through $x$.\[\]
Arrangements in higher dimensions
n hyperplanes in $\mathbb{R}^d$
- number of cells is $O(n^d)$
- zone thin: The zone of a hyperplane has complexity $O(n^{d-1})$

in 3D, n planes. There are $O(n^3)$ cells and a zone has complexity $O(n^2)$

Applications of Arrangements.

- Aspect graph
  - vertices = combinatorially distinct viewpoints of an object
  - edge = two viewpoints related by slight movement

See slide

The aspect graph can be used to
- find viewpoint seeing max. no. faces
- find "nice" viewpoint
- find where a robot is located based on what it sees
Aspect graph of a polygon

area with same viewpoint = cell in arrangement of lines through pairs of visible points

$O(n^2)$ lines, so $O(n^4)$ cells

Aspect graph of convex polyhedron with n vertices

$O(n)$ faces $\Rightarrow$ $O(n)$ planes $\Rightarrow$ $O(n^3)$ cells

For non-convex polyhedron

we need planes through every 3 points (in worst case)

so $O(n^3)$ planes $\Rightarrow$ $O(n^a)$ cells
Application - testing collinear points
(mentioned last day)

Given n points, are there 3 (or more) collinear?

Solution: apply duality point \( \leftrightarrow \) line
question becomes: are there 3 (or more)
lines through 1 point.
Construct the arrangement and test.
\( O(n^2) \).

Duality and arrangements

In dual:
cell = all lines
above points
above lines \( l' \), below \( l'' \)

Levels in an Arrangement

Assume no line is vertical.

Any vertical line \( l' \) not through
a vertex orders edges
intersecting \( l \)

Level \( L_1 = \) all edges topmost
for some \( l \)

Level \( L_i = \) all edges \( i^{th} \) in order for some \( l \)
Constructing levels in $O(n^2)$ time

Sort lines by slope.
Consider line $l$, we know its level at far left ($= \text{rank of slope}$).

$\xrightarrow{\text{level } i}$ $\rightarrow$ $\xrightarrow{\text{level } i-1}$ $\xrightarrow{\text{level } i}$ $\rightarrow$ $\xrightarrow{\text{level } i+1}$

Open problem on levels: What is the (worst case) number of edges in level $k$?
Dual: given a set of points, how many subsets of size $k$ can be cut away with a line?
For $k = \lfloor n/2 \rfloor$, how many halving lines can there be?

Best known bounds:
$\sum (n \log k)$
'73, Erdős et al.
raised a bit by Toth 2001
$O(n k^{1/3})$
Tamal Dey '97

Also: find level $k$ without constructing whole arrangement
A. Lubiw, U. Waterloo

Lecture 13: Arrangements

Figure 71: Narrowest corridor and trapezoidal maps.

Maximum Discrepancy:

Next we consider a problem derived from computer graphics and sampling. Suppose that we are given a collection of $n$ points $S$ lying in a unit square $U = [0, 1]^2$. We want to use these points for random sampling purposes. In particular, the property that we would like these points to have is that for any halfplane $h$, we would like the size of the fraction of points of $P$ that lie within $h$ should be roughly equal to the area of intersection of $h$ with $U$. That is, if we define $U(h)$ to be the area of $h$ intersection $U$, and $U_S(h) = |S \text{ intersection } h|/ |S|$, then we would like $U(h)$ about $U_S(h)$ for all $h$. This property is important when point sets are used for things like sampling and Monte Carlo integration.

Figure 72: Discrepancy of a point set.

To this end, we define the discrepancy of $S$ with respect to a halfplane $h$ to be

$$D_S(h) = |U(h) - U_S(h)|.$$ 

For example, in the figure, the area of $h$ intersect $U$ is $U(h) = 0.625$, and there are 7 out of 13 points in $h$, thus $U_S(h) = 7/13 = 0.538$. Thus the discrepancy of $h$ is $|0.625 - 0.538| = 0.087$. Define the halfplane discrepancy of $S$ to be the maximum (or more properly the supremum, or least upper bound) of this quantity over all halfplanes:

$$D_S = \sup h D_S(h).$$

Since there are an uncountably infinite number of halfplanes, it is important to derive some sort of finiteness criterion on the set of halfplanes that might produce the greatest discrepancy.

Lemma:

Let $h$ denote the halfplane that generates the maximum discrepancy with respect to $S$, and let $l$ denote the line that bounds $h$. Then either (i) $l$ passes through at least two points of $S$, or (ii) $l$ passes through one point of $S$, and this point is the midpoint of the line segment $l$ intersect $U$.

Remark:

If a line passes through one or more points of $S$, then should this point be included in $U_S(h)$? For nice presentation: [Link](http://www.ams.org/joyspi/features-col/col-2011-12)
Proof

If $h$ goes through no points we can slide up or down to increase discrepancy.

If $h$ goes through 1 point we can rotate up or down to increase discrepancy unless $p$ is mid-point.

Solving the discrepancy problem via arrangements:

Lines $h$ of type 2 can be checked brute-force.

For each point, it can be mid-point of only 2 segments. For each, count all pts. below.

$O(n^2)$.

Lines $h$ of type 1 — use dual arrangement:

- line $h$ through 2 points $\leftrightarrow$ point $h^*$ on 2 lines
- # points below $h \leftrightarrow$ # lines above $h^*$
- = level of $h^*$.

So we test all points of dual arrangement.

Each test is $O(1)$ (note: can compute area below $h$ in $O(1)$)

Total $O(n^2)$. 