Triangulations. Applications and criteria

- angle criteria - for meshing

- length criteria: minimum weight triangulation

- constrained triangulations (when certain edge must be included)

- meshing - triangulations with Steiner points

- flip distance

- morphing

- curve and surface reconstruction
Curve and surface reconstruction

Approximately 12 Minutes
Hundreds of Thousands Points Connected

Original \(\rightarrow\) Point Cloud \(\rightarrow\) 3D Model
Curve and surface reconstruction

digital Michaelangelo project
Curve and surface reconstruction

alpha-shapes and alpha-hulls

pushing lines against a point set gives the convex hull
line = infinite radius circle

pushing discs of smaller radius gives more refined “shape” and detects holes

the alpha-hull, alpha = disc radius
alpha-shapes and alpha-hulls

when alpha is small, the points remain isolated;
when alpha is large the alpha-hull approaches the convex hull
alpha-shapes and alpha-hulls

issues:
- what is the “right” value of alpha?
- if points are not uniform then no single value of alpha will work.

cited by 1939

Three-dimensional alpha shapes
Crust Algorithm for surface reconstruction

in 2D this is curve reconstruction

figures from Devadoss, O’Rourke

points on the curve must be sufficiently dense in order to reconstruct the curve


Delaunay triangulation of original points $S$ + Voronoi vertices

edges with both endpoints in $S$
Medial axis of a convex polygon = Voronoi diagram of edges of polygon

= locus of centers of circles inside polygon that touch boundary at 2 or more points
  (centers of maximal inscribed discs)
Medial axis of a convex polygon = Voronoi diagram of edges of polygon  

= grow the vertex angle bisectors by shrinking the polygon. The trajectories of the vertices form the medial axis.

every edge is a bisector of two polygon edges
Medial axis of a non-convex polygon = locus of centers of maximal inscribed discs

Figure 5.6: The central arc lies on the parabola determined by the vertex \( v \) and the edge \( e \), where the maximal disks centered on that arc touch \( e \) and \( v \).
a physical model for medial axis

Imagine the polygon is drawn on the prairie, and you light fires along the boundary. Medial axis = points where fire is quenched (fire meets other fire)

- pouring sand

Voronoi diagram
a physical model for medial axis
Applications of medial axis

Blum transform for shape recognition

character recognition

shape matching

Vadim Shapiro

Straight Skeleton — similar to medial axis but avoids curved sections

Grow the vertex angle bisectors by shrinking the polygon. The trajectories of the vertices form the straight skeleton.

same as medial axis for convex polygon

but not the same for a non-convex polygon:
Straight Skeleton — similar to medial axis but avoids curved sections

Difference between medial axis and straight skeleton — only for non-convex polygons:

offset curve with mitred caps
Straight skeleton algorithms

idea of previous algorithm gives $O(n^2 \log n)$ because the next ray intersection need not be between consecutive rays

improvements:

$O(n^{8/5} + \varepsilon)$ for any fixed $\varepsilon > 0$


$O(n^{4/3} + \varepsilon)$ time for any $\varepsilon > 0$

Straight skeleton applications: designing roofs

How to fit a roof to these walls?
Straight skeleton application: fold and cut problem

**Fold and Cut Theorem.** For any (slightly perturbed) polygon on a piece of paper there is a flat folding of the paper that puts all the polygon edges on one line and puts the inside and outside of the polygon on opposite sides of the line.
Straight skeleton application: fold and cut problem

**Fold and Cut Theorem.** For any (slightly perturbed) polygon on a piece of paper there is a flat folding of the paper that puts all the polygon edges on one line and puts the inside and outside of the polygon on opposite sides of the line.

![Fold and Cut Example](image)

solution for triangle:
What does a solution to fold and cut look like at the polygon boundary?

**MUST** use bisector at each vertex.
**MAY** use perpendiculars on any edge.

Thus, use straight skeleton.

**MUST** use angle bisector at each vertex.
Thus, use straight skeleton.

**MAY** use perpendiculars on any edge
and we need some of these to get flat folding.
Example.

All folds except the pink ones are straight skeleton folds.

In degenerate cases, this bouncing can be infinite. This is why we need to perturb input polygon.
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In degenerate cases, this bouncing can be infinite. This is why we need to perturb input polygon.
fold-and-cut examples


[http://erikdemaine.org/foldcut/](http://erikdemaine.org/foldcut/)
Adding line $l_i$ to the line arrangement:

To bound run time we need the Zone Theorem
Incremental Construction

Algorithm ConstructArrangement (L)

Input.
Set L of n lines

Output.
DCEL for A(L) in B(L)

1. Compute bounding box B(L)
2. Construct DCEL for subdivision induced by B(L)
3. for i = 1 to n
   do insert

\[ z(l_i) = 3+3+4+3+2+3 = 18 \]

zone of \( l_i \) has 6 faces

ignoring rectangular boundary