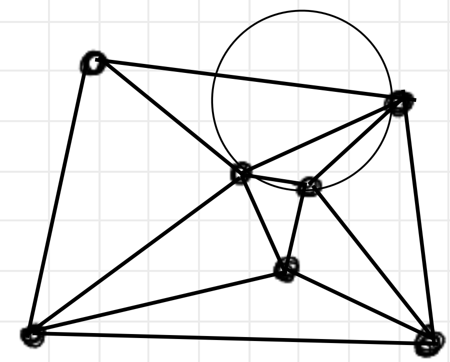
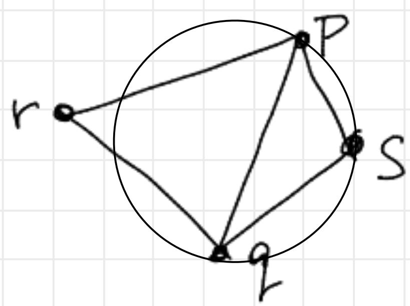
Recall Delaunay triangulation
and empty circle property: (p,q) is an edge of the Delaunay triangulation iff there is an empty circle through p and q.



An algorithmically more useful characterization:

Lemma. A triangulation is Delaunay iff for each edge e=(p,q) with triangles on each side, say pqr and pqs, r is not in circle(p,q,s).



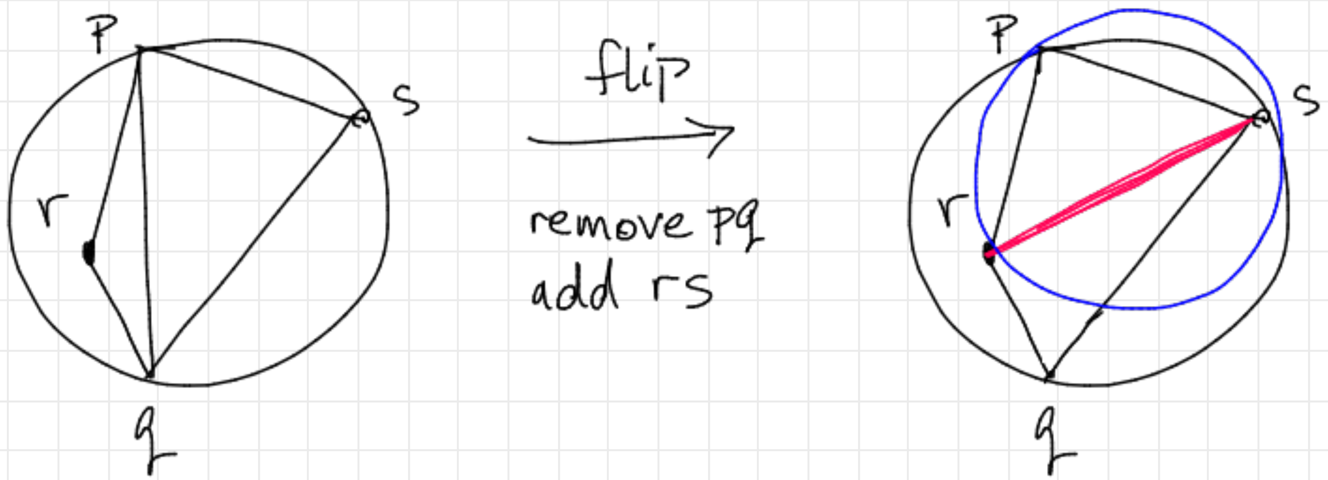
Note: r in circle(p,q,s) iff s in circle(p,q,r)

Call edge e *legal* if either:
- e is on the convex hull or
- e is interior and the above condition holds

proof of Lemma

What to do with an illegal edge (p,q)

Edge Flip



Claim: (r,s) is a legal edge.

Edge flips make global improvements in a triangulation: angle vector

Thus, flipping illegal edges always gets you to the Delaunay triangulation,
and the Delaunay triangulation has the lexicographically maximum angle vector.

Consequences:

Theorem. The Delaunay triangulation maximizes the minimum angle.

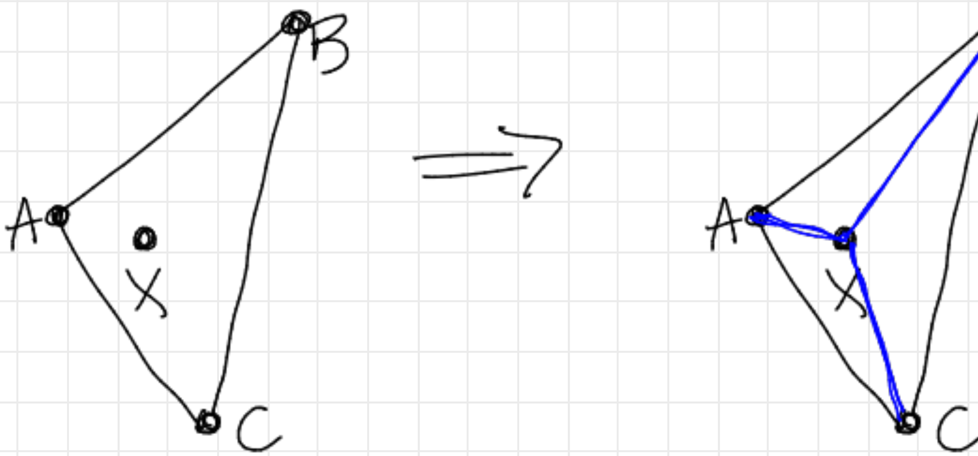
Algorithm to find the Delaunay triangulation: find ANY triangulation and then flip illegal edges until there are none.

How many flips does this take?

[Randomized] Incremental Delaunay triangulation algorithm.

to add a new point X:

Find current triangle ABC containing X

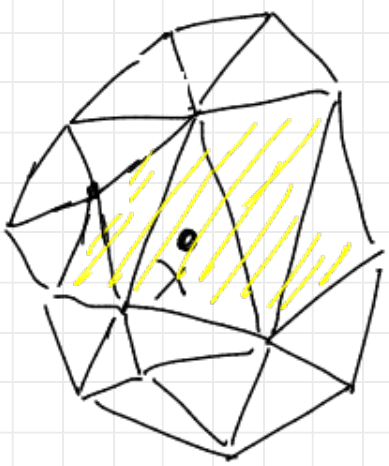


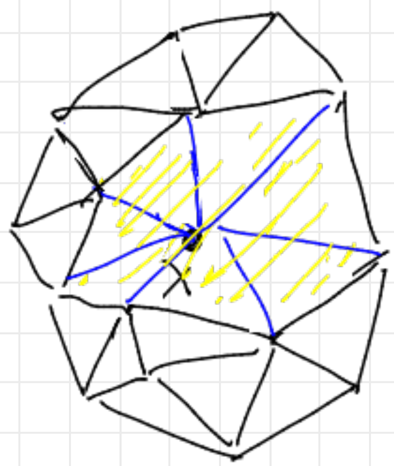
Join X to A, B, C

Then flip edges until Delaunay

issues, details

Changes produced by this Test update:





All the new edges will be incident to X.

correctness

Issues remaining:

1. how to find the triangle containing X

2. analysis of running time when points are inserted in random order.

What primitive operations are needed for this algorithm?

Given 4 points, A, B, C, D, is D inside Circle(A,B,C)?

By mapping last day

$$(x, y) \longrightarrow (x, y, z = x^2 + y^2)$$

this is the same as: is D below the plane through A, B, C?

This is a Sidedness test in 3D, and can be decided with a few multiplications, additions, subtractions.