Lemma. A triangulation is Delaunay iff for each edge \( e=(p,q) \) with triangles on each side, say \( pqr \) and \( pqs \), \( r \) is not in \( \text{circle}(p,q,s) \).

Note: \( r \) in \( \text{circle}(p,q,s) \) iff \( s \) in \( \text{circle}(p,q,r) \)

Call edge \( e \) *legal* if either:
- \( e \) is on the convex hull or
- \( e \) is interior and the above condition holds

**proof of Lemma**
What to do with an illegal edge \((p,q)\)

**Edge Flip**

\[
\text{flip} \quad \Rightarrow \quad \text{remove } pq \quad \text{add } rs
\]

Claim: \((r,s)\) is a legal edge.
Edge flips make global improvements in a triangulation: angle vector
Thus, flipping illegal edges always gets you to the Delaunay triangulation, and the Delaunay triangulation has the lexicographically maximum angle vector.

Consequences:

Theorem. The Delaunay triangulation maximizes the minimum angle.

Algorithm to find the Delaunay triangulation: find ANY triangulation and then flip illegal edges until there are none.

How many flips does this take?
[Randomized] Incremental Delaunay triangulation algorithm.

to add a new point $X$:

Find current triangle $ABC$ containing $X$

Then flip edges until Delaunay
issues, details
Changes produced by this Test update:

All the new edges will be incident to X.
Issues remaining:

1. how to find the triangle containing X

2. analysis of running time when points are inserted in random order.
What primitive operations are needed for this algorithm?

Given 4 points, A, B, C, D, is D inside Circle(A,B,C)?

By mapping last day

\[(x, y) \rightarrow (x, y, z = x^2 + y^2)\]

this is the same as: is D below the plane through A, B, C?

This is a Sidedness test in 3D, and can be decided with a few multiplications, additions, subtractions.