Recall

- every polygon can be triangulated
- there is an $O(n \log n)$ algorithm for polygons and polygonal regions
- this is best possible for polygonal regions
- polygons can be triangulated in O(n) via Chazelle's hard algorithm

Optimizing Polygon Triangulations

We may have criteria to prefer one triangulation over another.
For meshing, angles are often the main issue — more on this later.

Minimum Weight Triangulation ("minimum ink") — minimize the sum of the lengths of the chords used. (length = Euclidean length)

Polynomial Time Algorithm for Min Weight Triangulation of a Polygon (Klincsek 1980)

Idea: dynamic programming

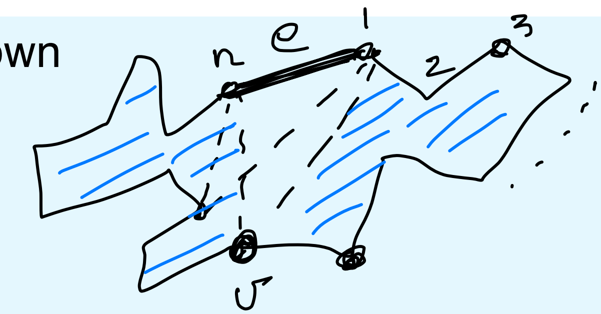
Input: Polygon P on vertices 1, 2, . . . , n

general steps for dynamic programming:

- identify subproblems & order of solving
- formula for solving subproblem in terms of smaller ones

Polynomial Time Algorithm for Min Weight Triangulation of a Polygon

think top-down



edge e is in a triangle with some 3rd vertex
– try all possibilities

subproblems

$\forall\, i<j \quad M_{ij} = $ min. length to triangulate subpolygon $i, i+1 \ldots j$

$-\infty$ if ij is not a chord.

$$= d_{ij} + \min_{k \; i<k<j} \{ M_{ik} + M_{kj} \}$$

d_{ij}: length of chord ij

ordering: by $j-i$ (smallest subpolygons first)

Run Time: $O(n^2)$ subproblems. $O(n)$ to solve one.

Total: $O(n^3)$

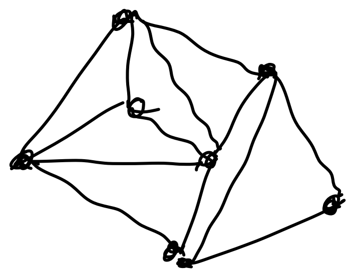
Polynomial Time Algorithm for Min Weight Triangulation of a Polygon

dynamic programming $O(n^3)$

Open Problem. $o(n^3)$ time algorithm for min weight triangulation.

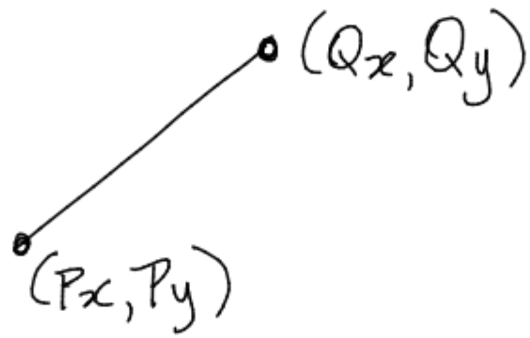
Min weight triangulation of a point set is NP-hard.

Minimum-weight triangulation is NP-hard
W Mulzer, G Rote - Journal of the ACM (JACM), 2008 - dl.acm.org
https://doi.org/10.1145/1346330.1346336



Real RAM Model of Computation

The above algorithm assumed we can compute Euclidean lengths and compare them at unit cost.



Euclidean length

$$\sqrt{(Q_x - P_x)^2 + (Q_y - P_y)^2}$$

We used a test of the form:

$$\sqrt{a_1} + \sqrt{a_2} + \cdots \sqrt{a_k} \overset{?}{\gtreqless} \sqrt{b_1} + \sqrt{b_2} + \cdots \sqrt{b_\ell}$$

a_i, b_i integer,

OPEN: Measuring bit complexity, can this test be done in polynomial time?

Real RAM Model of Computation

real RAM — random access machine that operates on real numbers.

Allows arithmetic, including square root, at unit cost.
Sometimes also allow k-th roots, trigonometric functions, etc.

This is a basic model of computation used in computational geometry.
In practice, we must address issues of precision.

For an interesting discussion of alternate models of computing for computational geometry, see section 1.2 in:

Finding Closed Quasigeodesics on Convex Polyhedra
ED **Demaine**, AC Hesterberg… - … Geometry (SoCG 2020), 2020 - drops.dagstuhl.de
https://drops.dagstuhl.de/opus/volltexte/2020/12191/

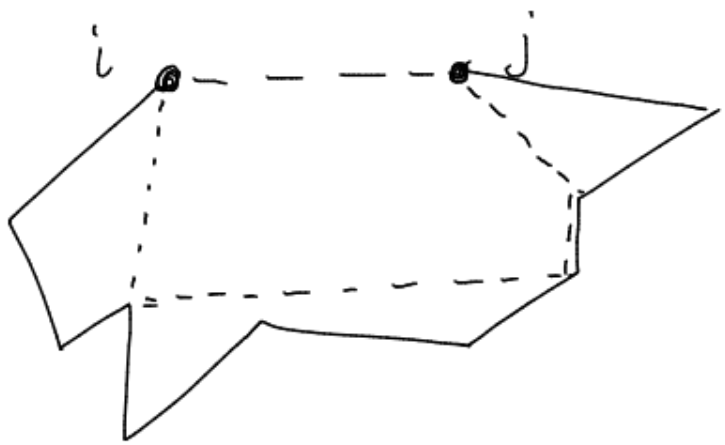
Partitioning Polygons into Convex Pieces

Sometimes we don't really need triangles — convex pieces will do.

Partitioning a polygon into a minimum number of convex pieces by adding chords.

$O(n^3 \log n)$ using dynamic programming. Keil 1985.

Decomposing a polygon into simpler components
JM **Keil** - SIAM Journal on Computing, **1985** - SIAM
https://doi.org/10.1137/0214056



Careful — there can be exponentially many subpolygons.

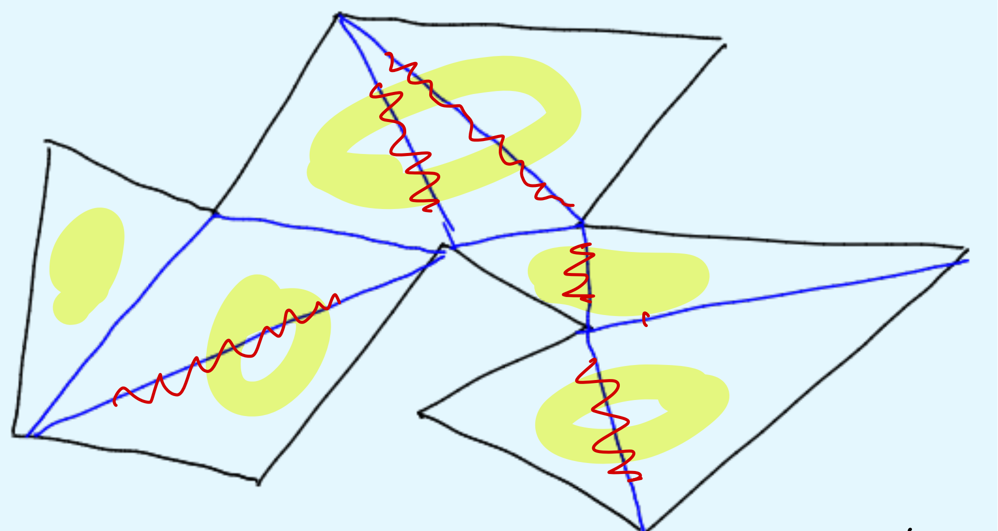
Partitioning Polygons into Convex Pieces

A faster approximation algorithm
Hertel and Mehlhorn 1983.

O(n log n) time and finds number of pieces ≤ 4 x min

Start with a triangulation

- remove any chord that leaves convex pieces

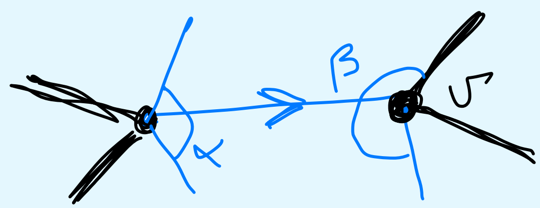


e.g. get 5 convex pieces here

<u>Claim</u>: #pieces ≤ 4·min

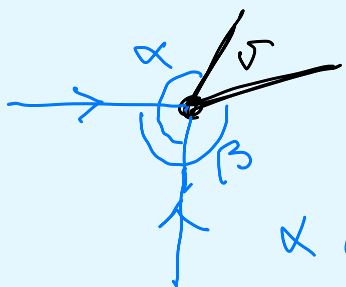
<u>Proof.</u>

#chords left = #pieces − 1



Any chord left (not removed) must leave a reflex angle (> 180°) at one side or the other.

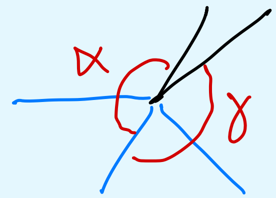
Claim: number of pieces is ≤ 4 x min



At vertex v, how many chords can "use" this vertex

At most 2.

Reason



$\alpha > 180$
$\gamma > 180$

impossible.

\# chords $\le 2r$
r = # reflex vertices of P
\# pieces $\le 2r+1$

On the other hand, every reflex vertex must have a chord and one chord can kill ≤ 2 reflex vertices

So min # chords $\ge \lceil \frac{r}{2} \rceil$, min # pieces $\ge \frac{r}{2}+1$

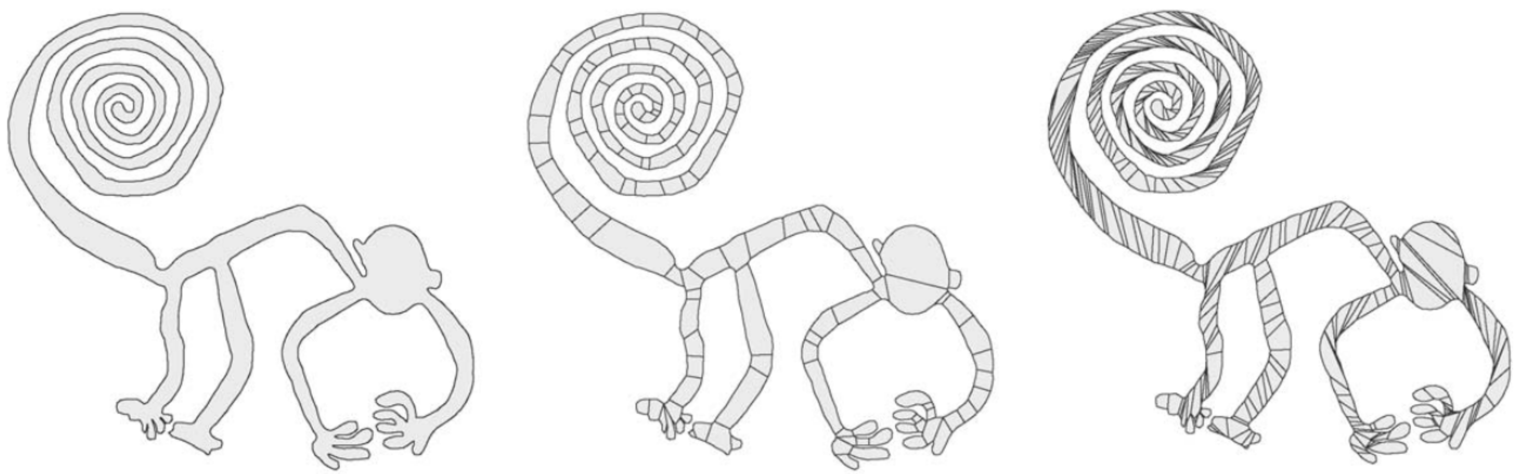
Thus 4 (min # pieces) $\ge 2r+4 \ge$ # pieces from alg.

Practical algorithms to decompose into approximately convex pieces

Approximate convex decomposition of polygons d https://doi.org/10.1016/j.comgeo.2005.10.005
JM Lien, NM **Amato** - Computational Geometry, 2006 - Elsevier
We propose a strategy to decompose a polygon, containing zero or more holes, into "approximately convex" pieces. For many applications, the approximately convex components of this decomposition provide similar benefits as convex components, while the …

Approximate convex decomposition of polyhedra and its applications d https://doi.org/10.1016/j.cagd.2008.05.003
JM Lien, NM **Amato** - Computer Aided Geometric Design, 2008 - Elsevier
Decomposition is a technique commonly used to partition complex models into simpler components. While decomposition into convex components results in pieces that are easy to . . .

J.-M. Lien, N.M. Amato / Computational Geometry 35 (2006) 100–123 101



(a) (b) (c)

Fig. 1. (a) The initial Nazca monkey has 1,204 vertices and 577 notches. The radius of the minimum bounding circle of this model is 81.7 units. Setting the concavity tolerance at 0.5 units, and not allowing Steiner points, (b) an approximate convex decomposition has 126 approximately convex components, an (c) a minimum convex decomposition has 340 convex components.

Dissecting one polygon into another

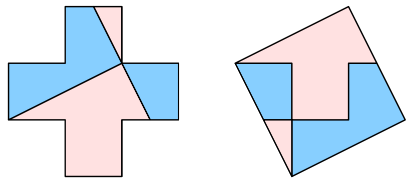


Figure 1: 4-piece dissection of Greek cross to square from 1890 [25].

Any two polygons of the same area have a common dissection (1807).

https://en.wikipedia.org/wiki/Bolyai%E2%80%93Gerwien_theorem

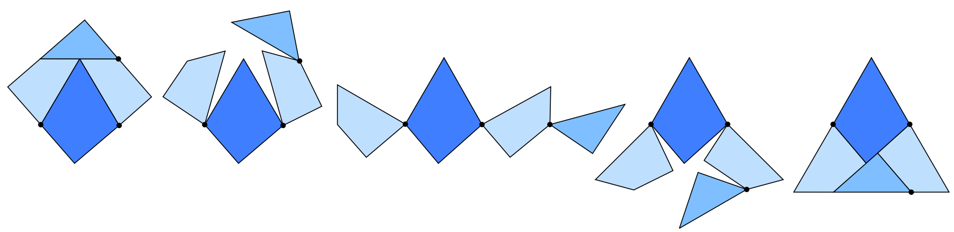


Figure 2: Dudeney's 1902 hinged dissection of a square into a triangle [15].

Hinged dissections exist https://doi.org/10.1007/s00454-010-9305-9

TG Abbott, Z Abel, D Charlton, ED Demaine… - Discrete & …, 2012 - Springer

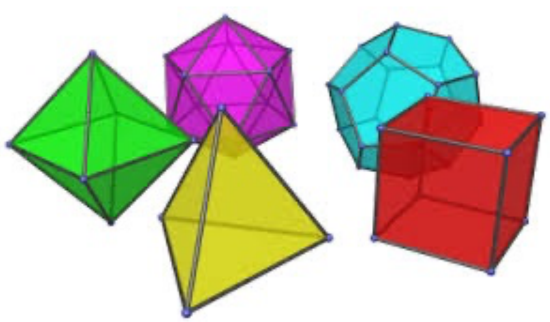
We prove that any finite collection of polygons of equal area has a common hinged dissection. That is, for any such collection of polygons there exists a chain of polygons hinged at vertices that can be folded in the plane continuously without self-intersection to …

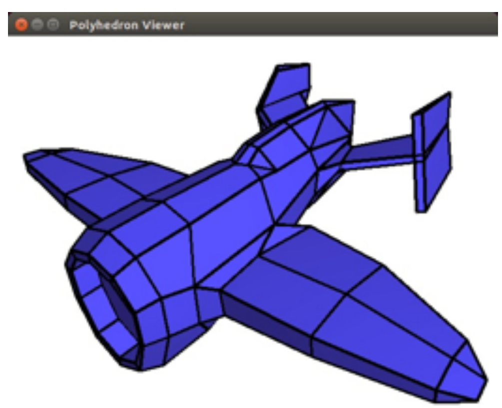
Polyhedra

A ***polyhedron*** consists of a finite connected set of (plane) polygons called ***faces*** such that

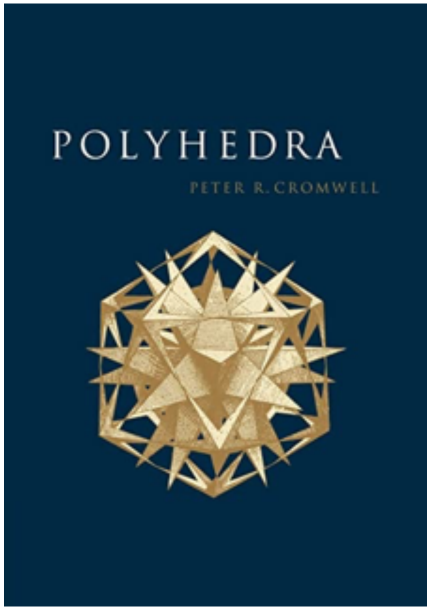
1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

W https://en.wikipedia.org/wiki/Polyhedron



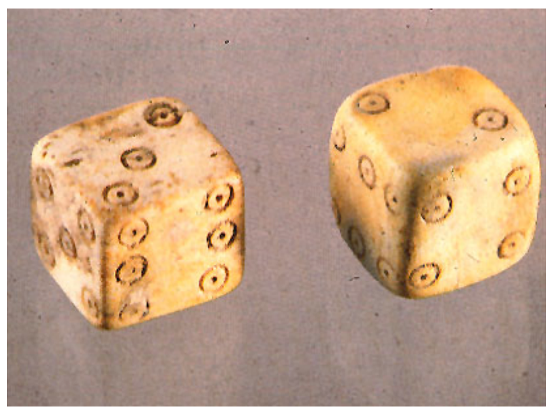


https://doc.cgal.org/latest/Polyhedron/index.html



https://tinyurl.com/yy5tt439

Polyhedra

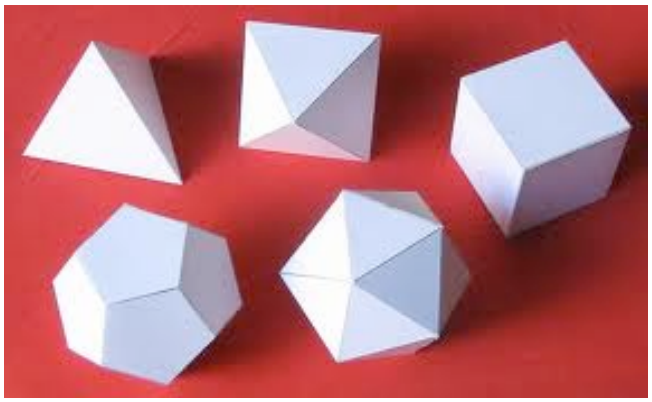


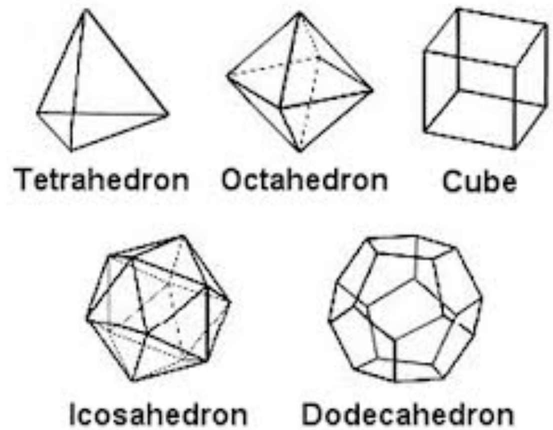
dice, Pompei, 1st century

icosahedral die, Roman, 2nd century

Polyhedra

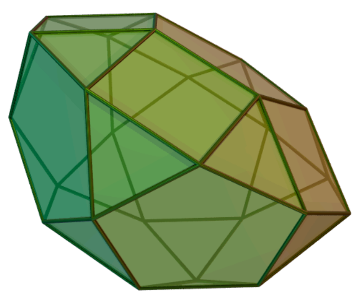
Platonic solids



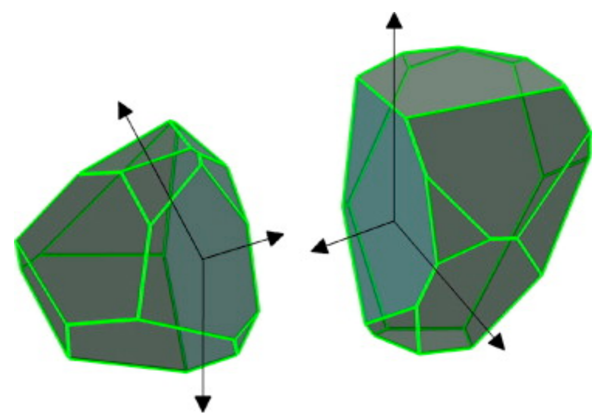


nets

cuboctahedron



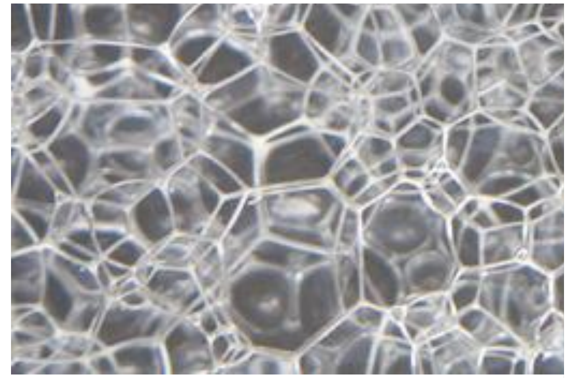
Pentagonal orthocupolarotunda



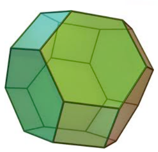
polycrystalline morphology

Lord Kelvin's Bubble Problem

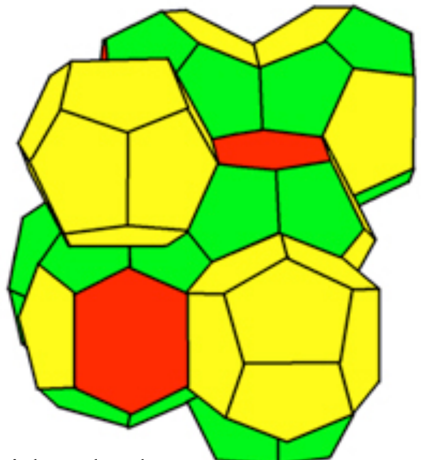
Cells of equal volume with minimum surface area



Kelvin structure, 1887
(truncated octahedra)



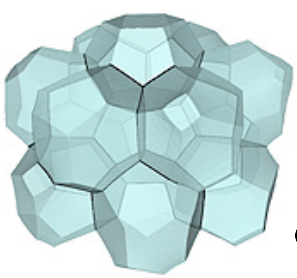
Weaire-Phelan structure, 1993
W https://en.wikipedia.org/wiki/Weaire–Phelan_structure



tetrakaidecahedron

irregular dodecahedron

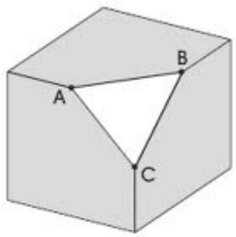
2008 Beijing Olympics

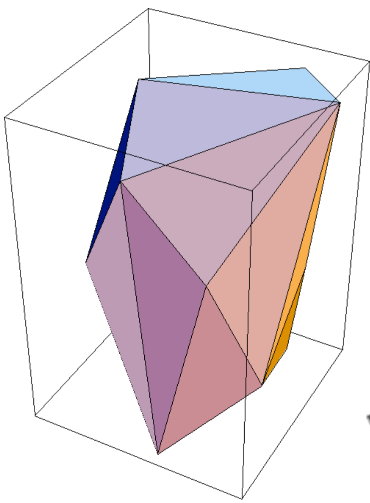


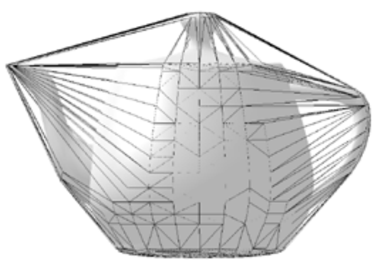
Gabrielli's structure, 2009

Non-convex Polyhedra

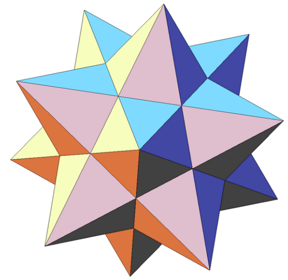
convex

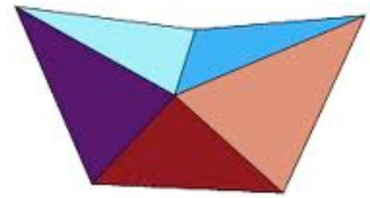


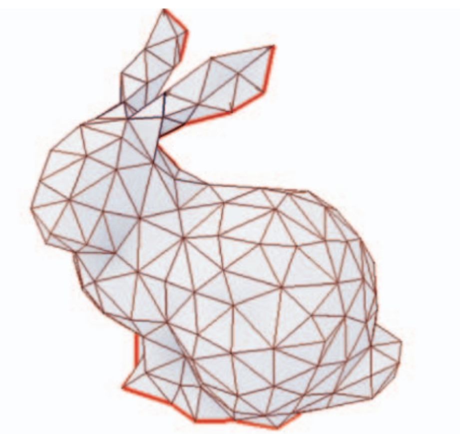




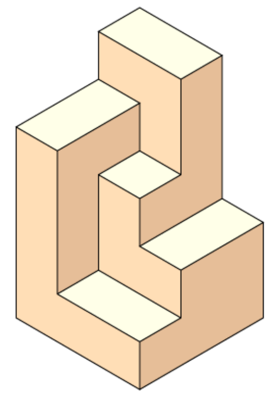
non-convex



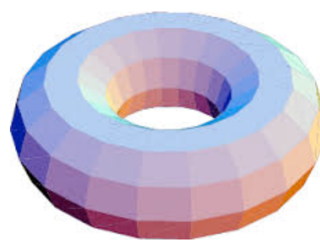


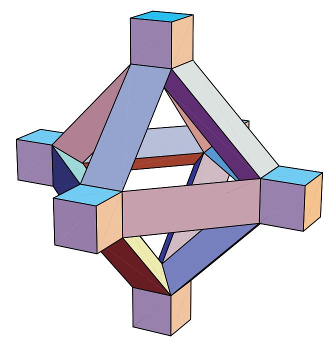


Tomohiro Tachi



David Eppstein



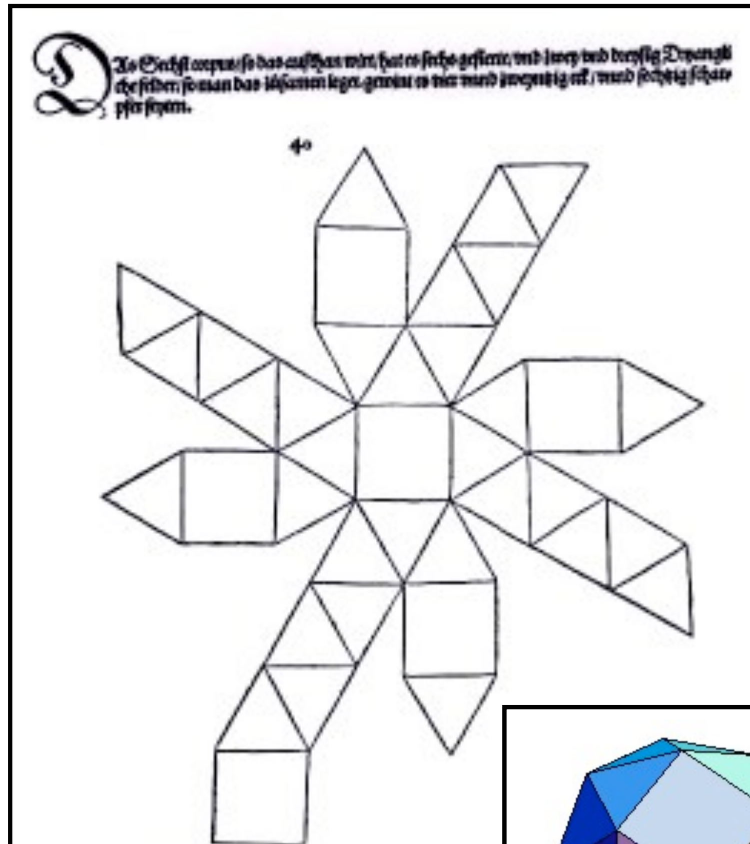


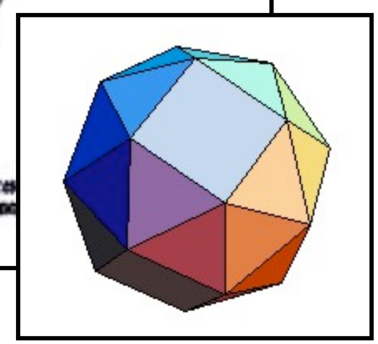
Donoso & O'Rourke

Maybe later in the course we will talk about unfolding polyhedra.

Unfolding Polyhedra—Durer 1400's

Durer, 1498



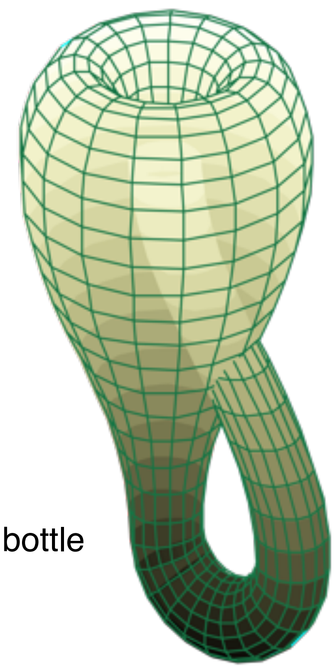


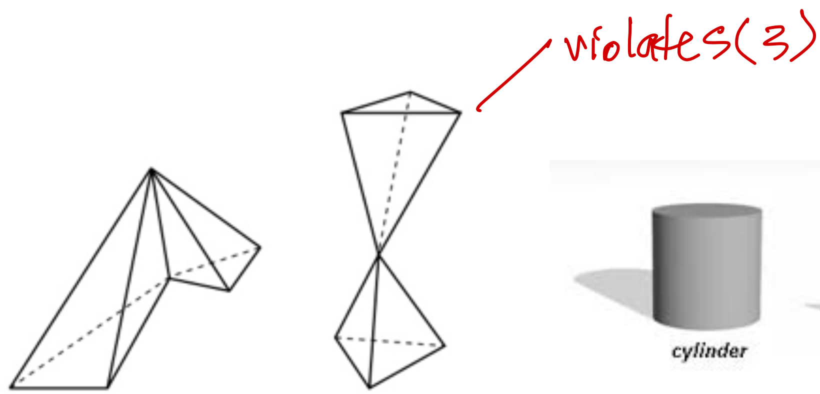
snub cube

A ***polyhedron*** consists of a finite connected set of (plane) polygons called ***faces*** such that

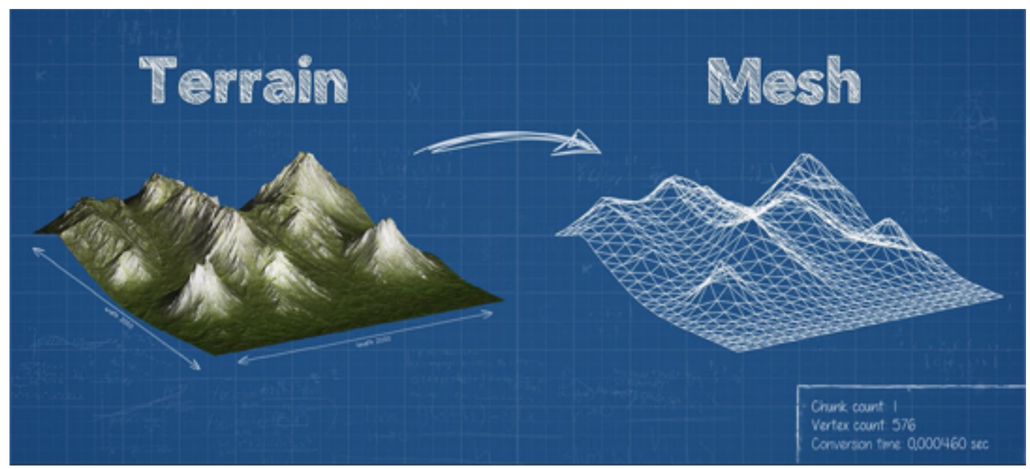
1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

NOT Polyhedra


Klein bottle



terrain

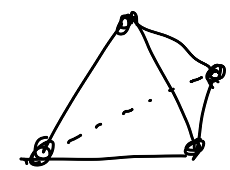


Mesh Materializer

A ***polyhedron*** consists of a finite connected set of (plane) polygons called ***faces*** such that

1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

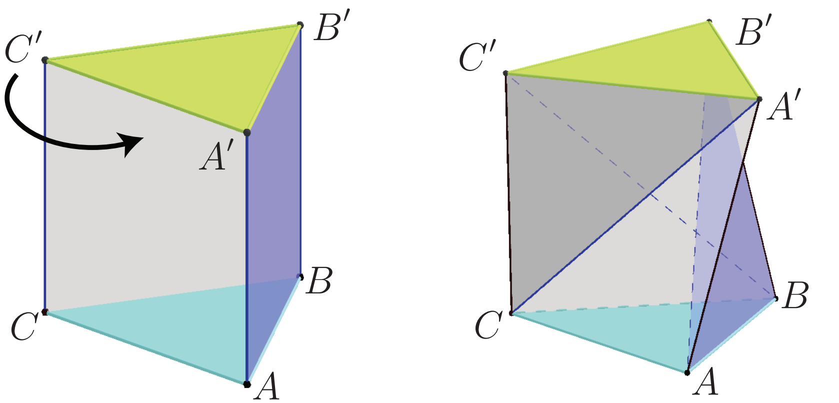
A ***tetrahedron*** is a polyhedron with 4 triangular faces. (aka a *simplex*)



To ***tetrahedralize*** a polyhedron means to partition its interior into disjoint tetrahedra whose vertices are vertices of the polyhedron.

Not all polyhedra can be tetrahedralized

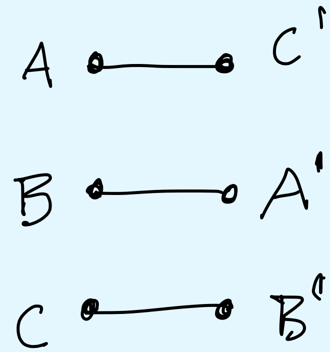
Schönhardt 1928



triangular prism with top face twisted, produces reflex edge in each rectangular face

why no tetrahedralization?

each vertex forbids one other vertex A forbid c′



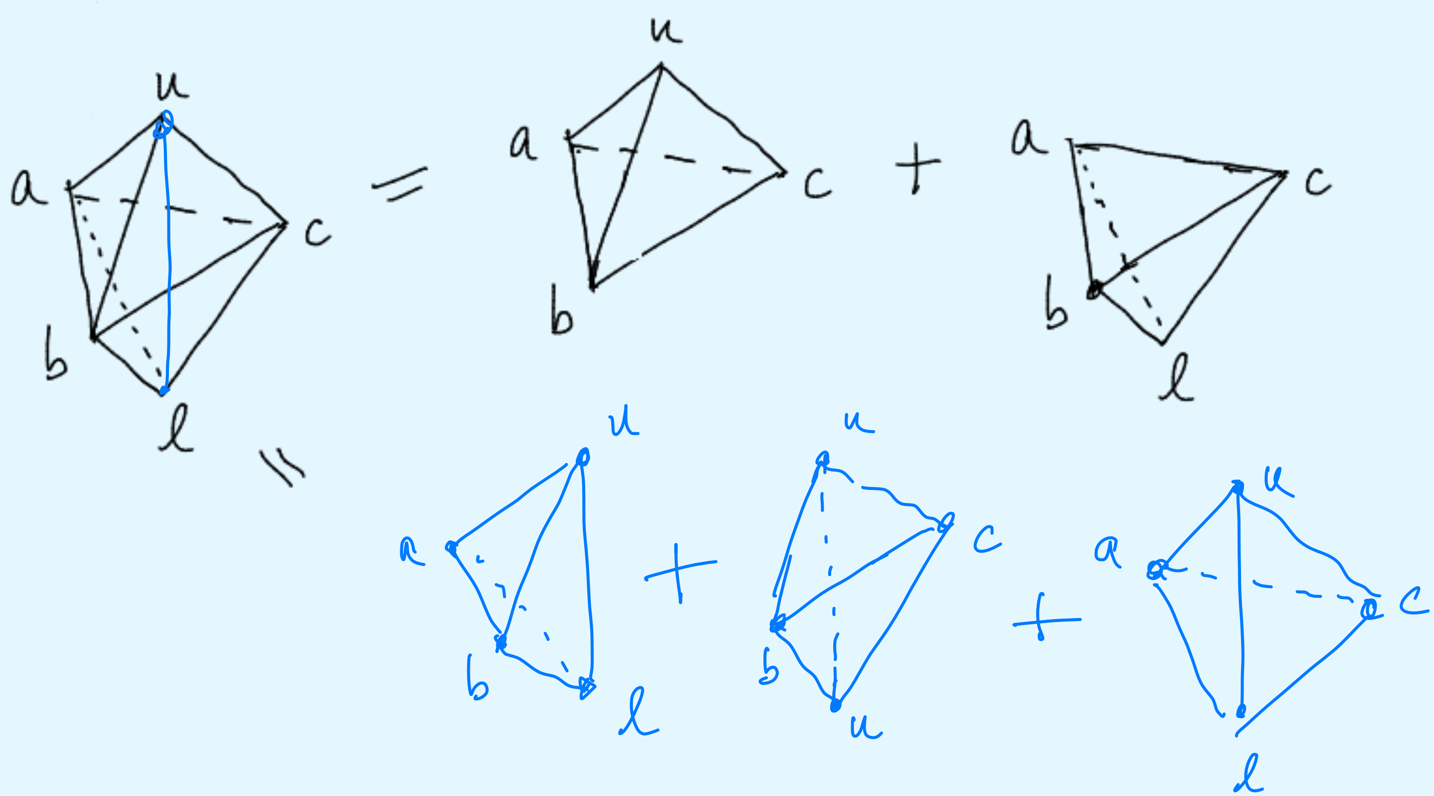
We want 4 vertices, no two forbidden.

impossible — no ind. set of 4 vertices.

On the difficulty of triangulating three-dimensional nonconvex polyhedra
J **Ruppert**, R **Seidel** - Discrete & Computational Geometry, 1992 - Springer https://doi.org/10.1007/BF02187840
A number of different polyhedral decomposition problems have previously been studied, most notably the problem of triangulating a simple polygon. We are concerned with the polyhedron triangulation problem: decomposing a three-dimensional polyhedron into a set …

The number of tetrahedra in a tetrahedralization is not unique

Example:



Exercise: Show that a cube can be cut into 5 tetrahedra and into 6 tetrahedra.

There are examples where number of tetra. can be $2n-7$ or $\binom{n-2}{2}$

Using Steiner points to partition a polyhedron into tetrahedra

Note: the output is no longer combinatorial — we need coordinates for Steiner points

Determining the min number of Steiner points for a given polyhedron is NP-hard.

Determining the minimum number of tetrahedra for a given polyhedron is NP-hard.
Even for convex polyhedra! (where min. number of Steiner points is 0)

https://link.springer.com/content/pdf/10.1007/s004540010058.pdf
Minimal simplicial dissections and triangulations of convex 3-polytopes
A Below, U Brehm, JA De Loera… - Discrete & Computational …, 2000 - Springer

d https://doi.org/10.1016/S0196-6774(03)00092-0
The complexity of finding small triangulations of convex 3-polytopes
A Below, JA De Loera, J Richter-Gebert - Journal of Algorithms, 2004 - Elsevier

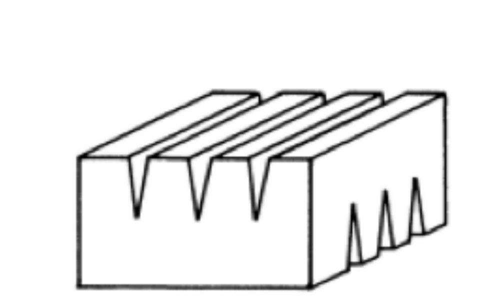
Can adding Steiner points reduce the number of tetrahedra? Yes.

Exercise: Find an example.

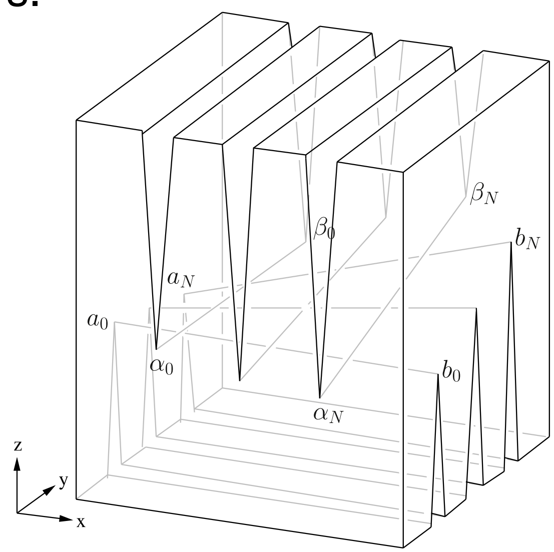
Explore efficient algorithms to approximate the min number of Steiner points or tetrahedra to within some guaranteed ratio.

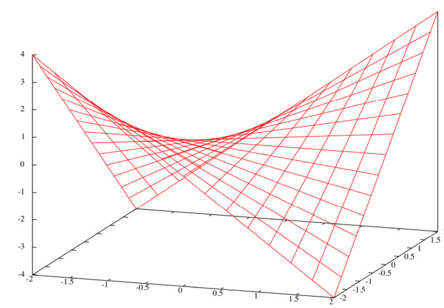
Using Steiner points to partition a polyhedron into tetrahedra

a lower bound:
There are polyhedra that require Omega(n^2) Steiner points even to partition into convex pieces. Chazelle, 1980's.



Chazelle





Hang Si & Nadja Goerigk

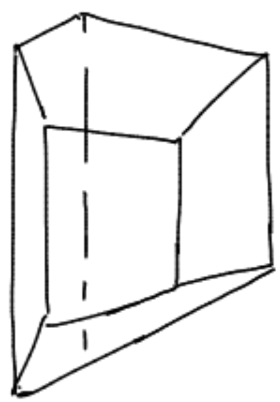
Cut wedges from a cube so they *almost* meet in the middle, and their lines form a hyperbolic paraboloid. The lines cut the hyperbolic paraboloid into Theta(n^2) pieces, pairwise invisible, so Omega(n^2) convex pieces are needed in any partition.

d https://doi.org/10.1137/0213031
Convex partitions of polyhedra: a lower bound and worst-case optimal algorithm
B **Chazelle** - SIAM Journal on Computing, 1984 - SIAM
The problem of partitioning a polyhedron into a minimum number of convex pieces is known to be NP-hard. We establish here a quadratic lower bound on the complexity of this problem, and we describe an algorithm that produces a number of convex parts within a constant …

a positive result:
Any polyhedron can be partitioned into O(n^2) tetrahedra using O(n^2) Steiner points.

Bern and Eppstein, “Mesh generation and optimal triangulation”, 1995

Idea — a bit like trapezoidization:
- from each edge of the polyhedron, extend a vertical wall up and down.
- pieces are “generalized prisms”



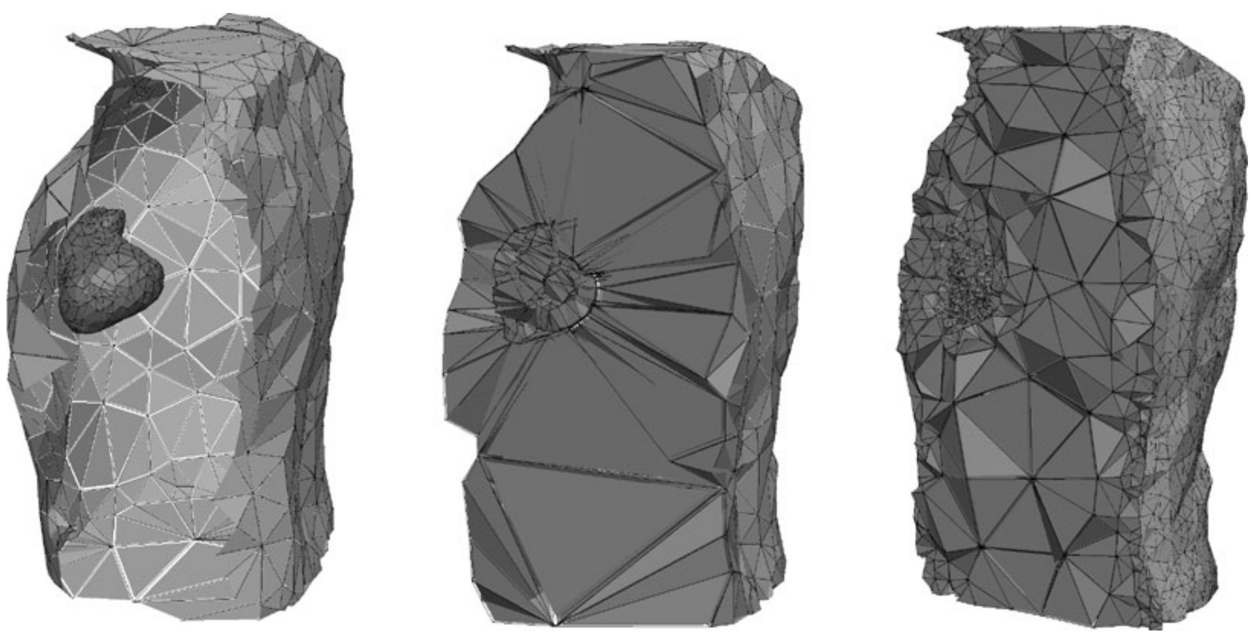
- vertical sides (each is a trapezoid)
- one top face, one bottom face
  (not necessarily parallel)

- this gives O(n^2) pieces
- then tetrahedralize these pieces:
    - cut into triangular prisms by triangulating the top and bottom the same way
    - then add one Steiner point in each, making sure that tetrahedra match face-to-face

an approach from meshing (uses Delauany tetrehedralization, which we'll cover later on)

https://doi.org/10.1007/3-540-29090-7_9

Meshing piecewise linear complexes by constrained Delaunay tetrahedralizations
H Si, K Gärtner - Proceedings of the 14th international meshing …, 2005 - Springer
We present a method to decompose an arbitrary 3D piecewise linear complex (PLC) into a constrained Delaunay tetrahedralization (CDT).



Summary

- optimizing triangulations, dynamic programming technique
- partitioning polygons into convex pieces
- polyhedra
- partitioning polyhedra

References

- [O'Rourke] 2.5
- [Handbook] Chapter 30

Further topic

We only discussed partitioning. What about covering (the pieces are allowed to overlap), or Boolean combinations?