Recall

- every polygon can be triangulated
- there is an $O(n \log n)$ algorithm for polygons and polygonal regions
- this is best possible for polygonal regions
- polygons can be triangulated in $O(n)$ via Chazelle’s hard algorithm
Optimizing Polygon Triangulations

We may have criteria to prefer one triangulation over another. For meshing, angles are often the main issue — more on this later.

**Minimum Weight Triangulation** ("minimum ink") — minimize the sum of the lengths of the chords used. (length = Euclidean length)

Polynomial Time Algorithm for Min Weight Triangulation of a Polygon (Klinčsek 1980)

Idea: dynamic programming

Input: Polygon P on vertices 1, 2, . . . , n

general steps for dynamic programming:
- identify subproblems & order of solving
- formula for solving subproblem in terms of smaller ones
Polynomial Time Algorithm for Min Weight Triangulation of a Polygon

think top-down

subproblems

\[ \forall \; i < j : \; M_{ij} = \begin{cases} \min \text{ length to triangulate subpolygon i, i+1 \ldots j } \\ -\infty \; \text{ if } ij \text{ is not a chord.} \end{cases} \]

\[ = d_{ij} + \min_{k \; i < k < j} \{ M_{ik} + M_{kj} \} \]

ordering: by \( j-i \) (smallest subpolygons first)

Run Time: \( O(n^2) \) subproblems, \( O(n) \) to solve one.

Total: \( O(n^3) \)
Polynomial Time Algorithm for Min Weight Triangulation of a Polygon

dynamic programming $O(n^3)$

Open Problem. $o(n^3)$ time algorithm for min weight triangulation.

Min weight triangulation of a point set is NP-hard.

Minimum-weight triangulation is NP-hard
W Mulzer, G Rote - Journal of the ACM (JACM), 2008 - dl.acm.org
https://doi.org/10.1145/1346330.1346336
Real RAM Model of Computation

The above algorithm assumed we can compute Euclidean lengths and compare them at unit cost.

$\sqrt{(Q_x-P_x)^2 + (Q_y-P_y)^2}$

We used a test of the form:

$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_k} \overset{?}{\leq} \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_k}$

$a_i, b_i$ integer.

OPEN: Measuring bit complexity, can this test be done in polynomial time?
Real RAM Model of Computation

real RAM — random access machine that operates on real numbers.

Allows arithmetic, including square root, at unit cost. Sometimes also allow k-th roots, trigonometric functions, etc.

This is a basic model of computation used in computational geometry. In practice, we must address issues of precision.

For an interesting discussion of alternate models of computing for computational geometry, see section 1.2 in:

Finding Closed Quasigeodesics on Convex Polyhedra
ED Demaine, AC Hesterberg... - ... Geometry (SoCG 2020), 2020 - drops.dagstuhl.de
https://drops.dagstuhl.de/opus/volltexte/2020/12191/
Partitioning Polygons into Convex Pieces

Sometimes we don’t really need triangles — convex pieces will do.

Partitioning a polygon into a minimum number of convex pieces by adding chords.

$O(n^3 \log n)$ using dynamic programming. Keil 1985.

Careful — there can be exponentially many subpolygons.
Partitioning Polygons into Convex Pieces

A faster approximation algorithm
Hertel and Mehlhorn 1983.

O(n log n) time and finds number of pieces ≤ 4 x min

Start with a triangulation
- remove any chord that leaves convex pieces

Claim: # pieces ≤ 4 x min

Proof:
# chords = # pieces - 1 left

e.g. get 5 convex pieces here

Any chord left (not removed) must leave a reflex angle (> 180°) at one side or the other.
Claim: number of pieces is \( \leq 4 \times \min \# \text{ pieces} \)

At vertex \( v \), how many chords can "use" this vertex

At most 2.

Reason: 

\( r \) and \( \beta \) are reflex

\( \# \text{ chords} \leq 2r \)

\( r = \# \text{ reflex vertices of } P \)

\( \# \text{ pieces} \leq 2r + 1 \)

On the other hand, every vertex must have a chord and one chord can kill \( \leq 2 \) reflex vertices.

So \( \min \# \text{ chords} \geq \left\lceil \frac{r}{2} \right\rceil \), \( \min \# \text{ pieces} \geq \frac{r}{2} + 1 \)

Thus

\[ 4 \left( \min \# \text{ pieces} \right) \geq 2r + 4 \geq \# \text{ pieces from alg}. \]
Practical algorithms to decompose into approximately convex pieces

Approximate convex decomposition of polygons  
JM Lien, NM Amato - Computational Geometry, 2006 - Elsevier
We propose a strategy to decompose a polygon, containing zero or more holes, into “approximately convex” pieces. For many applications, the approximately convex components of this decomposition provide similar benefits as convex components, while the …

Approximate convex decomposition of polyhedra and its applications  
JM Lien, NM Amato - Computer Aided Geometric Design, 2008 - Elsevier
Decomposition is a technique commonly used to partition complex models into simpler components. While decomposition into convex components results in pieces that are easy to . . .

Fig. 1. (a) The initial Nazca monkey has 1,204 vertices and 577 notches. The radius of the minimum bounding circle of this model is 81.7 units. Setting the concavity tolerance at 0.5 units, and not allowing Steiner points, (b) an approximate convex decomposition has 126 approximately convex components, an (c) a minimum convex decomposition has 340 convex components.
Dissecting one polygon into another

Any two polygons of the same area have a common dissection (1807).

Figure 1: 4-piece dissection of Greek cross to square from 1890 [25].

Figure 2: Dudeney’s 1902 hinged dissection of a square into a triangle [15].

https://en.wikipedia.org/wiki/Bolyai%E2%80%93Gerwien_theorem

Hinged dissections exist. https://doi.org/10.1007/s00454-010-9305-9

TG Abbott, Z Abel, D Charlton, ED Demaine ... - Discrete & ..., 2012 - Springer

We prove that any finite collection of polygons of equal area has a common hinged dissection. That is, for any such collection of polygons there exists a chain of polygons hinged at vertices that can be folded in the plane continuously without self-intersection to ...
Polyhedra

A *polyhedron* consists of a finite connected set of (plane) polygons called *faces* such that

1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

W: https://en.wikipedia.org/wiki/Polyhedron

https://doc.cgal.org/latest/Polyhedron/index.html
Polyhedra

dice, Pompei, 1st century
icosahedral die, Roman, 2nd century
Polyhedra

Platonic solids

- Tetrahedron
- Octahedron
- Cube
- Icosahedron
- Dodecahedron

nets

cuboctahedron
Pentagonal orthocupolarotunda
polycrystalline morphology
**Lord Kelvin’s Bubble Problem**

Cells of equal volume with minimum surface area

Kelvin structure, 1887 (truncated octahedra)

Weaire-Phelan structure, 1993


Gabrielli’s structure, 2009

irregular dodecahedron

tetrakaidecahedron

2008 Beijing Olympics
Non-convex Polyhedra

convex
don-convex

A. Lubiw, U. Waterloo
Maybe later in the course we will talk about unfolding polyhedra.

**Unfolding Polyhedra—Durer 1400’s**

![Durer, 1498](image1)

![snub cube](image2)
A **polyhedron** consists of a finite connected set of (plane) polygons called **faces** such that
1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

**NOT Polyhedra**
A polyhedron consists of a finite connected set of (plane) polygons called faces such that

1. if two faces intersect it is only at a common vertex or edge
2. every edge of every face is an edge of exactly one other face
3. the faces surrounding each vertex form a single circuit

A tetrahedron is a polyhedron with 4 triangular faces. (aka a simplex)

To tetrahedralize a polyhedron means to partition its interior into disjoint tetrahedra whose vertices are vertices of the polyhedron.
Our Results.

and (c), and also in Figure 2. An example of two triangles with no banded surface is shown in Figure 3(a).

Each quadrilateral layers.

points) at a succession of

may lead to more general solutions where we design

most of our results we concentrate on the case where no Steiner points are allowed. Understanding this case

in

the polygon are maintained throughout this morph in the sense that an edge may become a polygonal path

intersection of the surface with the plane

monotone means that the surface provides a morph from

to

by 60

The best bound we know on the number of Steiner points required for a banded surface is

. (The fact that these cones do not self-intersect is proved in [9].) The condition that the surface is

reflex edge in each

triangular prism with top face twisted,

produces reflex edge in each rectangular face

why no tetrahedralization?

each vertex forbids one other vertex A forbid c'

A' - c'  we want

4 vertices,

B' - A'  no two

forbidden -

impossible - no ind. set

of 4 vertices.

triangular prism with top face twisted, produces reflex edge in each rectangular face

On the difficulty of triangulating three-dimensional nonconvex polyhedra

J Ruppert, R Seidel - Discrete & Computational Geometry, 1992 - Springer

A number of different polyhedral decomposition problems have previously been studied,

most notably the problem of triangulating a simple polygon. We are concerned with the

polyhedron triangulation problem: decomposing a three-dimensional polyhedron into a set …
The number of tetrahedra in a tetrahedralization is not unique

Example:

Exercise: Show that a cube can be cut into 5 tetrahedra and into 6 tetrahedra.

There are examples where number of tetra. can be $2n-7$ or $\binom{n-2}{2}$.
Using Steiner points to partition a polyhedron into tetrahedra

Note: the output is no longer combinatorial — we need coordinates for Steiner points

Determining the min number of Steiner points for a given polyhedron is NP-hard.

Determining the minimum number of tetrahedra for a given polyhedron is NP-hard. Even for convex polyhedra! (where min. number of Steiner points is 0)

Can adding Steiner points reduce the number of tetrahedra? Yes.

Exercise: Find an example.

Explore efficient algorithms to approximate the min number of Steiner points or tetrahedra to within some guaranteed ratio.
Using Steiner points to partition a polyhedron into tetrahedra

a lower bound:
There are polyhedra that require $\Omega(n^2)$ Steiner points even to partition into convex pieces. Chazelle, 1980’s.

Cut wedges from a cube so they
almost meet in the middle, and their lines form a hyperbolic paraboloid. The lines cut the hyperbolic paraboloid into $\Theta(n^2)$ pieces, pairwise invisible, so $\Omega(n^2)$ convex pieces are needed in any partition.

Convex partitions of polyhedra: a lower bound and worst-case optimal algorithm
B Chazelle - SIAM Journal on Computing, 1984 - SIAM
The problem of partitioning a polyhedron into a minimum number of convex pieces is known to be NP-hard. We establish here a quadratic lower bound on the complexity of this problem, and we describe an algorithm that produces a number of convex parts within a constant …
a positive result:
Any polyhedron can be partitioned into $O(n^2)$ tetrahedra using $O(n^2)$ Steiner points.
Bern and Eppstein, “Mesh generation and optimal triangulation”, 1995

Idea — a bit like trapezoidization:
- from each edge of the polyhedron, extend a vertical wall up and down.
- pieces are “generalized prisms”
- vertical sides (each is a trapezoid)
- one top face, one bottom face
  (not necessarily parallel)

- this gives $O(n^2)$ pieces
- then tetrahedralize these pieces:
  - cut into triangular prisms by triangulating the top and bottom the same way
  - then add one Steiner point in each, making sure that tetrahedra match face-to-face
an approach from meshing (uses Delaunay tetrahedralization, which we’ll cover later on)

Meshing piecewise linear complexes by constrained Delaunay tetrahedralizations
H Si, K Gärtner - Proceedings of the 14th international meshing ..., 2005 - Springer
We present a method to decompose an arbitrary 3D piecewise linear complex (PLC) into a constrained Delaunay tetrahedralization (CDT).
Summary

- optimizing triangulations, dynamic programming technique
- partitioning polygons into convex pieces
- polyhedra
- partitioning polyhedra

References

- [O’Rourke] 2.5
- [Handbook] Chapter 30

Further topic

We only discussed partitioning. What about covering (the pieces are allowed to overlap), or Boolean combinations?