Recall Every simple polygon can be triangulated.

Following the proof yields an obvious $O(n^4)$ time algorithm, which can be improved to $O(n^2)$.

Today: practical $O(n \log n)$ time algorithm.

History:

- 1978. First $O(n \log n)$ algorithm. Garey, Johnson, Preparata, Tarjan.
- 1984. Simpler. Fournier and Montuno. — this is what we'll study
- . . . $O(n \log \log n)$ . . . $O(n \log^* n)$ . . .
- 1991. $O(n)$ algorithm. Chazelle. But it is too complicated to implement. (Uses polygon-cutting theorem, planar separator theorem, but no fancy data structures.)

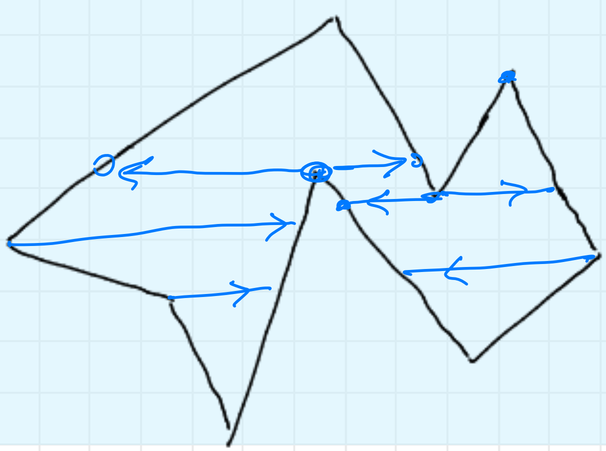
There is also an $O(n \log^* n)$ randomized algorithm by Seidel.

Triangulation Algorithm

Assume points have distinct y coordinates (else imagine tipping slightly).

Note: degeneracy and precision are big topics — we may cover some.

Step 1. Find a ***trapezoidization*** of the polygon — from each vertex shoot a horizontal line inside the polygon until it hits the boundary.



This ~~divid~~ partitions the polygon into trapezoids (each with horizontal top & bottom)

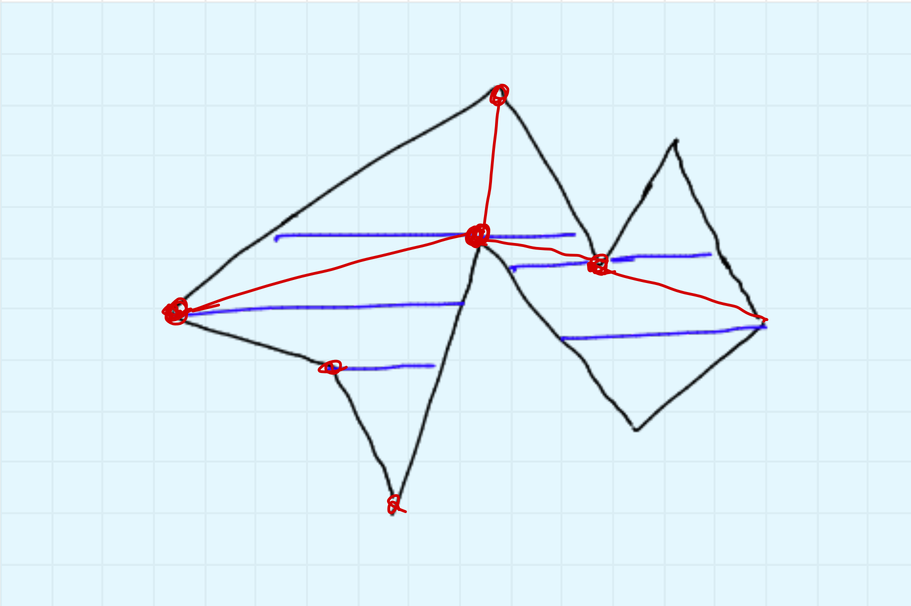
Note that a triangle is a degenerate trapezoid.

Step 2. From the trapezoidization, compute a triangulation.

Step 2 takes O(n) time.
Step 1, using plane-sweep, takes O(n log n).
Other algorithms make step 1 faster.

Step 2. From trapezoids to triangles.

First join trapezoids to form ***unimonotone*** polygons.



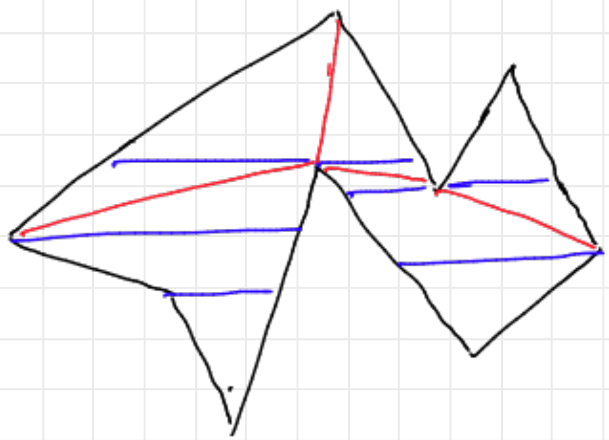
Every trapezoid has a vertex on the bottom and on the top (exactly one, since we assumed distinct y-coordinates).

If these are not joined by an edge, then add a chord.

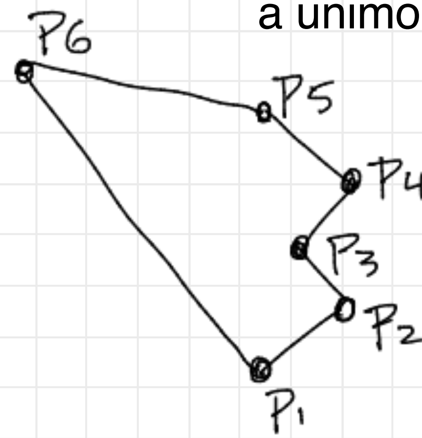
Step 2. From trapezoids to triangles.

definition

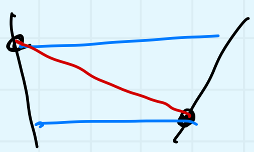
Resulting pieces are ***unimonotone*** polygons — with vertices in order of y-coordinate.



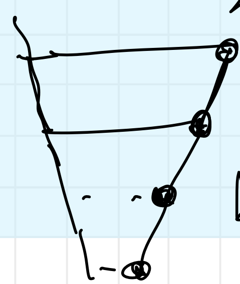
a unimonotone polygon



Proof.



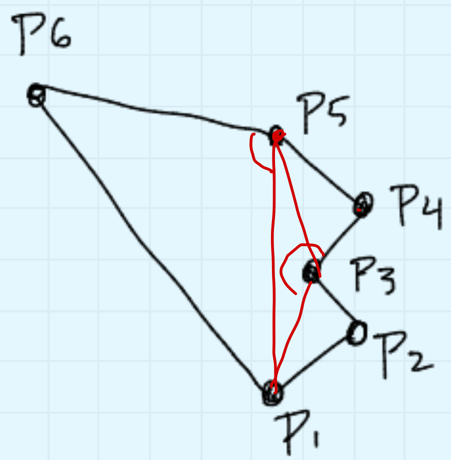
Every trapezoid not cut by a red chord is like this



and can only be followed (above/below) by another with both vertices on same side

Triangulating a unimonotone polygon in linear time.

Any convex vertex (except p_1, p_n) provides an ear, and cutting off the ear leaves a unimonotone polygon. Note that there is such a convex vertex.




```
v := p3
loop
    while prev(v) ≠ p1 and prev(v) convex

        cut off the ear at prev(v)
        update prev( ), next( ) pointers
        EXIT if the polygon is now empty

    v := next(v)
```

Correctness: by induction

RunTime: $O(n)$

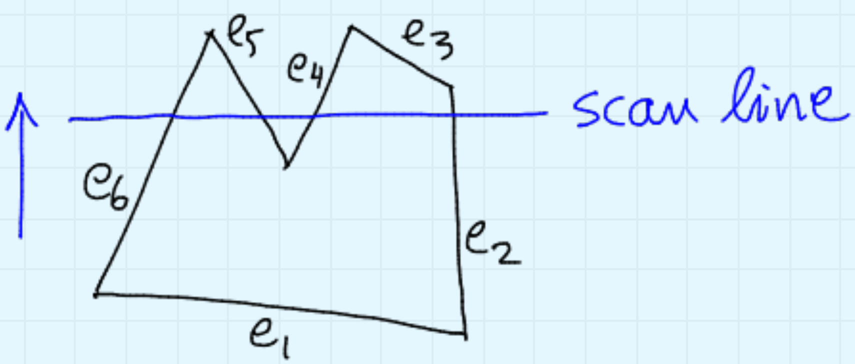
Step 1. Trapezoidization.

Use Plane-Sweep, a basic technique in planar computational geometry.
First used by Bentley and Ottman, 1979 to find intersections of line segments in the plane.

Plane Sweep (as a general paradigm)

Sweep a horizontal scan line across the plane, from bottom to top, analyzing the sequence of 1-dimensional cross-sections and the changes to them.

Cross-section = list of edges that cross the scan line (left to right)



cross-section is ordered list

— e_6 e_5 e_4 e_2

The cross section only changes at vertices.

Plane Sweep Algorithm (for non crossing segments)

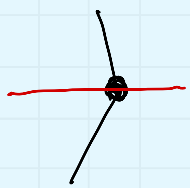
order vertices by y coordinate (assume distinct)

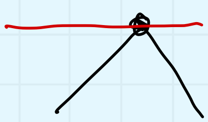
initialize cross-section = ∅ (at y = –∞)

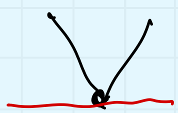
for each vertex p in order

update cross-section at p

possible updates:

1.  one edge replaces another

2.  two edges deleted

3.  two edges added

Plane Sweep Algorithm (for non crossing segments)

how to update the cross-section at p

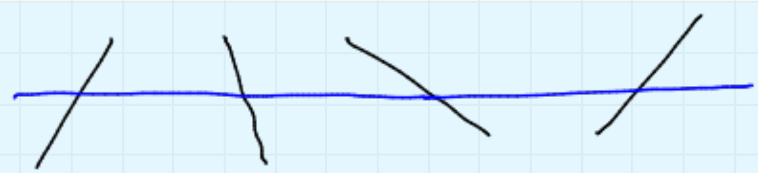
locate p in the current cross section

determine which situation (1, 2, or 3) applies and perform appropriate update

---

how to locate p in the current cross section

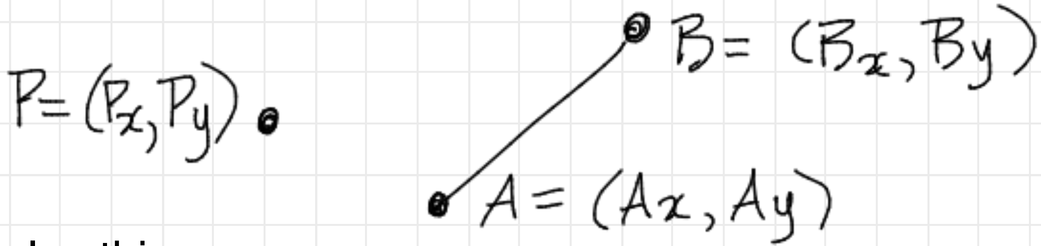
the cross section contains edges ordered by x
(we will store these in a balanced search tree)



We need an elementary test:

Does p lie left/right of edge e

Elementary test needed by Plane Sweep:
Sidedness Test: Input: 3 points, A, B, P. Is P on/left/right of line through A and B?



How to solve this:

- compute equation of line $y = ax + b$? – division by 0
  – precision issues

Test signed area of $\Delta PAB = \mathcal{A}$

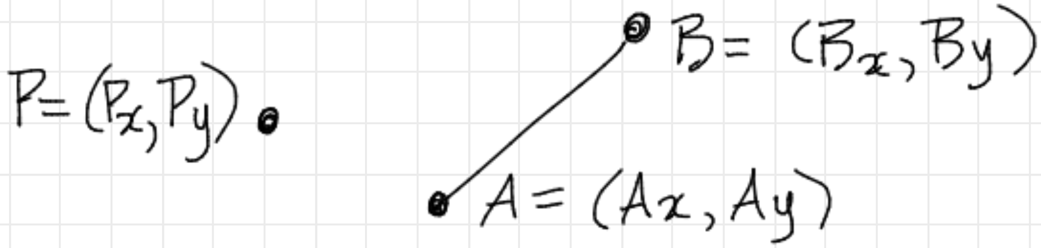
$2\mathcal{A}$ = cross-product of $B - A$ and $P - A$

$$= (B_x - A_x)(P_y - A_y) - (P_x - A_x)(B_y - A_y)$$

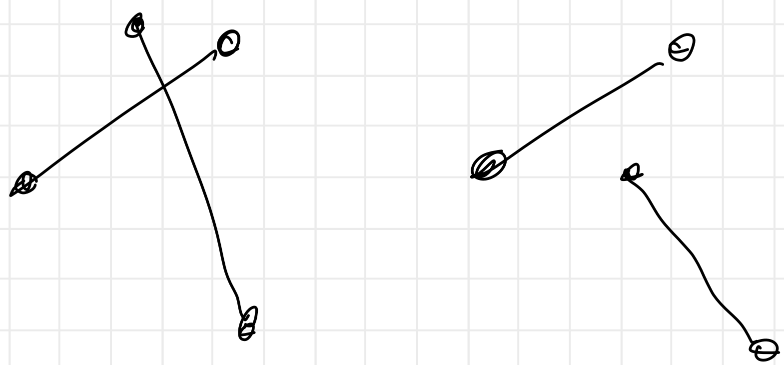
Note: integer arithmetic (if input is integers)
two multiplications

$$P \text{ left of } AB,\ \text{right},\ \text{on} \iff 2\mathcal{A} > 0,\ 2\mathcal{A} < 0,\ = 0$$

Elementary test needed by Plane Sweep:
Sidedness Test: Input: 3 points, A, B, P. Is P on/left/right of line through A and B?



Exercise: Use the sidedness test to test if two line segments intersect.
Note how this avoids special cases for vertical lines, parallel lines, etc.



Data structure for Plane Sweep

maintain ordered list (the list of edges crossing the scan line) to allow
find, insert, delete

use balanced search tree

AVL trees, red/black -

$O(\log n)$ per find/insert/delete.

Timing for Plane Sweep

sort $O(n \log n)$

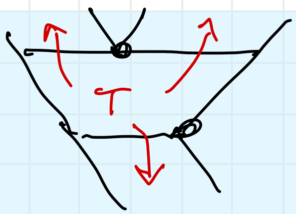
+ n updates at $O(\log n)$ each

$O(n \log n)$

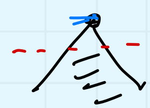
Details of plane-sweep for trapezoidization.

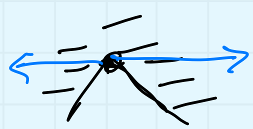
As the cross section is updated, collect information on each trapezoid

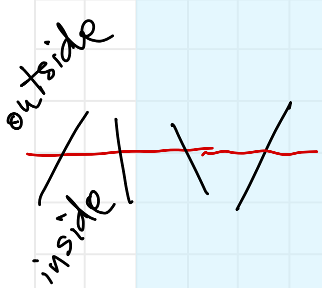
- left & right polygon edges, bottom and top coordinates
- bottom and top neighbours (note 0, 1, or 2 of each)

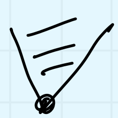


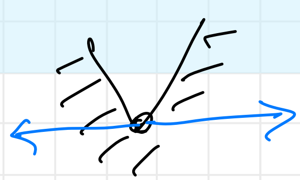
1.  one trapezoid ends, one begins

2. a)  one trapezoid end

b)  two trapezoids end, one begins

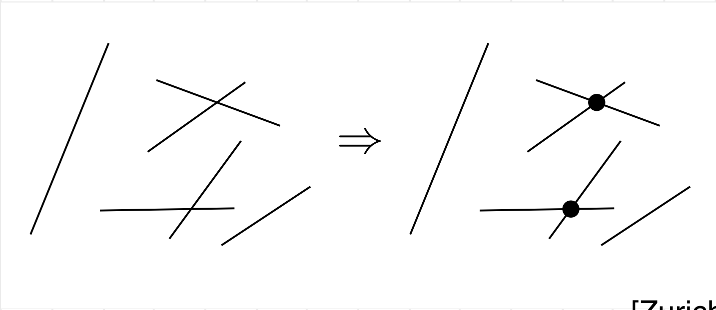


3. (a)  one trapezoid begins

b)  one trapezoid ends, two begin

Applications of general plane-sweep

- find all k intersections of n line segments — there may be Theta(n^2)



[Zurich notes]

k = # intersection points

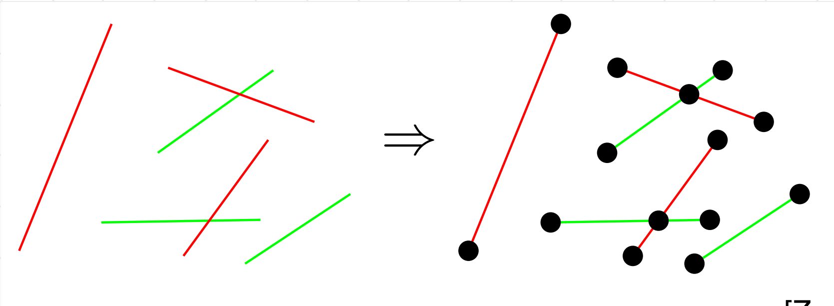
general plane sweep can solve this in $O(n \log n + k \log n)$ time.

Robust plane sweep for intersecting segments https://doi.org/10.1137/S0097539797329373
JD Boissonnat, FP Preparata - SIAM Journal on Computing, 2000 - SIAM

Crossing patterns in nonplanar road networks https://doi.org/10.1145/3139958.3139999
D Eppstein, S Gupta - Proceedings of the 25th ACM SIGSPATIAL …, 2017 - dl.acm.org

Applications of general plane-sweep

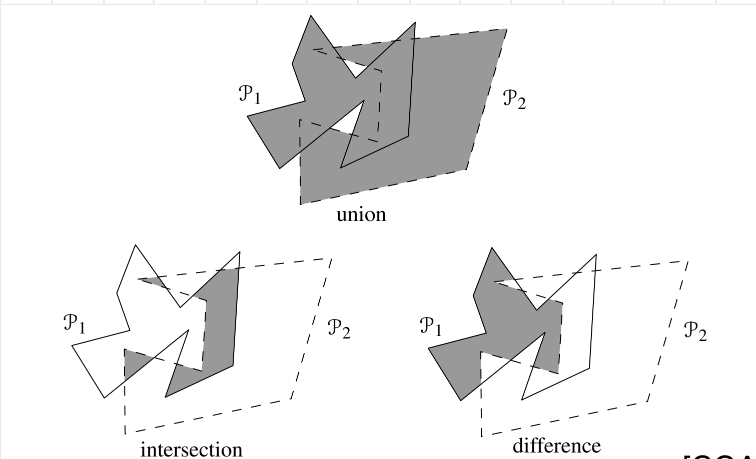
- map overlay



[Zurich notes]

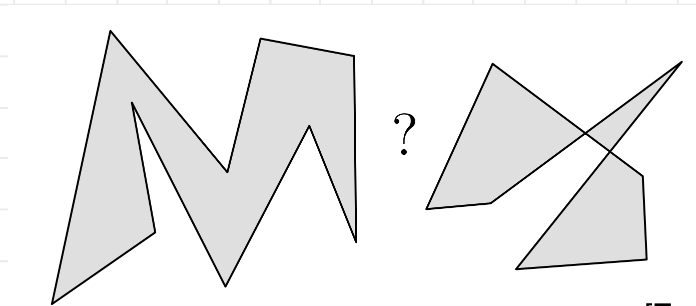
[Overlaying simply connected planar subdivisions in linear time](https://doi.org/10.1145/220279.220292) https://doi.org/10.1145/220279.220292
U **Finke**, KH **Hinrichs** - Proceedings of the eleventh annual symposium …, 1995 - dl.acm.org

- Boolean operations on polygons



[CGAA]

- Is a polygon simple?



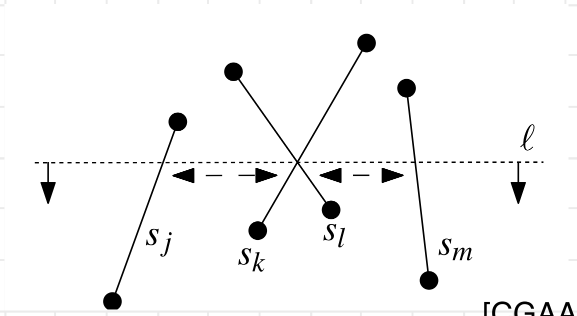
[Zurich notes]

$O(n \log n)$

O(n) by Chazelle

Sorry, I forgot to cover this slide!

General Plane Sweep — segments may cross
(The above plane sweep assumed that segments do not cross.)



[CGAA]

The order of segments changes at an intersection point — we must add the intersection point as an "event" (in addition to the endpoints).

How do we find these events? By checking pairs of adjacent edges in cross-sections.
Use a Priority Queue (PQ) to store events.

initialize cross-section = ∅ (at $y = -\infty$)
initialize PQ to contain vertices
while PQ not empty
    get next event from Priority Queue
    update cross-section at the event
    if a new pair of edges becomes consecutive in the cross section, test
        for intersection and add event

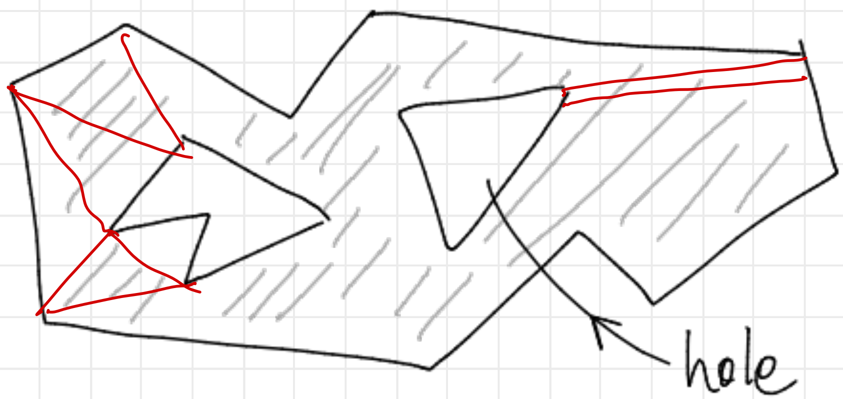
$O(n \log n + k \log n)$ k = number of intersections (might be n^2)
See [CGAA], or [Zurich Notes] — discusses primitives

improvement to $O(n \log n + k)$ (hard)
further improvement to $O(n)$ space

d https://doi.org/10.1145/220279.220302
An optimal algorithm for finding segments intersections
IJ Balaban - Proceedings of the eleventh annual symposium on …, 1995 - dl.acm.org

Triangulating a polygonal region.

More general than polygon — *polygonal region* = polygon with holes



Plane Sweep for non-crossing segments still works. O(n log n)
(Note that we did not require connectedness of the boundary.)

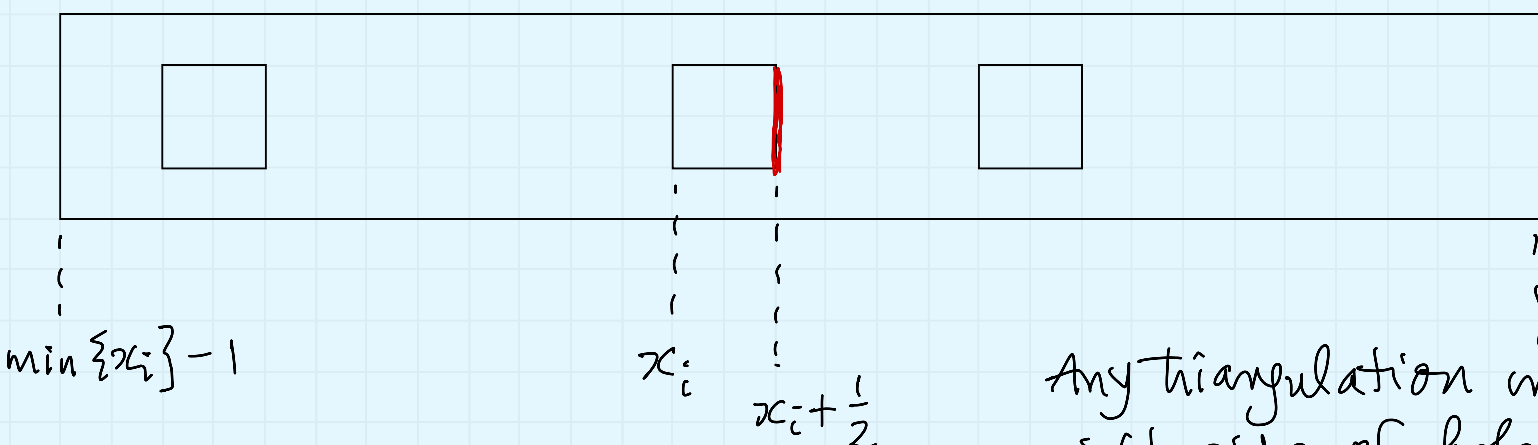
But faster algorithms (e.g. Chazelle's linear time algorithm) do not work.

A lower bound for triangulating a polygonal region.

Asano, Asano, Pinter, 1986.
Idea: the problem is as hard as sorting, so requires Omega($n \log n$).
Must be careful of the model of computation.

Reduce sorting to triangulation.
Given n distinct integers $x_1, x_2, \ldots x_n$ to sort, construct a polygonal region such that triangulating gives the sorted order.

polygonal region = rectangle with n square holes



Any triangulation must link right side of hole for x_i to left side of next x_j.
Follow links to get sorted order

A lower bound for triangulating a polygonal region.

Note that this requires indirect addressing.

Thus, we need a stronger model of computation than the comparison-based model where sorting has an easy Omega(n log n) lower bound.

Model: unit cost RAM (Random Access Machine) with

- indirect addressing
- branching based on comparisons
- arithmetic +, - , x

There is an Omega(n log n) lower bound for sorting on this model (Paul and Simon, 1982).

Thus triangulating polygonal region takes $\Omega(n \log n)$ on this model

Summary

- plane sweep algorithm (basic tool in 2D)
- O(*n* log *n*) triangulation for polygons and polygonal regions
- pay attention to basic steps of geometric algorithms – sidedness test
- pay attention to model of computing for lower bounds

References

- [CGAA] Sections 3.2, 3.3 (slightly different algorithm)
- [Zurich notes] Appendix A
- [O'Rourke] 2.1 - 2.4