Recall a problem we considered before: given n points, are there 3 (or more) collinear.

By duality (points $\leftrightarrow$ lines) this becomes: given n lines, do 3 of them intersect at a point.

To get an $O(n^2)$ algorithm, we study **line arrangements**.

A set of n lines in the plane partitions the plane into faces (cells), edges, vertices, called the **arrangement**.
Recall

How to update after adding line \( l_i \) to the line arrangement:

To bound the run time we need the Zone Theorem

**Definitions.** Let \( A \) be an arrangement, and \( l \) be a line not in \( A \). The **zone** of \( l \) in arrangement \( A \) is \( Z_A(l) = \{ \text{faces of } A \text{ cut by } l \} \).

The **size** of the zone is \( z_A(l) = \sum \{ \# \text{ edges in face } f : f \in Z_A(l) \} \).

\[ z_n = \max \{ z_A(l) : \text{over all possible} \ l, \ A \text{ of } n \text{ lines} \} \]

**Zone Theorem.** \( z_n \) is \( O(n) \).
Arrangements in higher dimensions

For an arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \)

- the number of cells is \( O(n^d) \)

- Zone Theorem. The zone of a hyperplane has complexity \( O(n^{d-1}) \)

In 3D, for \( n \) planes, there are \( O(n^3) \) cells, and a zone has complexity \( O(n^2) \).
Application: Aspect Graph

What are all the combinatorially distinct viewpoints of an object?

Figure 7: Aspect graph of a cube. The front, left, right, back, top and under sides of the cube are denoted by the letters F, L, R, B, T and U respectively.

http://im-possible.info/english/articles/animation/animation.html
Application: Aspect Graph

Area with the same viewpoint = cell in arrangement of lines through pairs of visible points.

$n^2$ lines, so $n^4$ cells.

Can we really get $\Omega(n^2)$ lines for a convex polygon — $\Theta(n)$ lines?
Application: Aspect Graph

Aspect graph of convex polyhedron in 3D with n vertices

\[ O(n) \text{ faces} \Rightarrow O(n^3) \text{ cells} \]

for non-convex polyhedron \( \Theta(n^9) \) cells

we need planes through every 3 points (in worst case)
so \( O(n^3) \) planes \( \Rightarrow \) \( O(n^9) \) cells

The aspect graph can be used to:

- find the viewpoint seeing the maximum number of faces
- find a “nice” projection
- figure out where a robot is, based on what it sees.
Recall Duality Map

point in $\mathbb{R}^2$ $\quad \leftrightarrow \quad$ line in $\mathbb{R}^2$

$p = (p_1, p_2)$ $\quad \leftrightarrow \quad$ $y = p_1 x - p_2$ (not defined for vertical lines)

Lemma. point $p$ lies on/above/below line $q^*$ iff point $q$ lies on/above/below line $p^*$.

Corollary. Some points lie on a line iff their dual lines go through a point.
Properties of duality map

Lemma. point p lies on/above/below line q* iff point q lies on/above/below line p*.

Corollary. Some points lie on a line iff their dual lines go through a point.

Proof. \( \begin{align*}
p &= (a, b) \\
q &= (c, d)
\end{align*} \)

\( \begin{align*}
p^* : y &= ax - b \\
q^* : y &= cx - d
\end{align*} \)

above
\( p \) lies on \( q^* \)
above
\( q \) lies on \( p^* \)

\( b = ca - d \)
\( d = ac - b \)

\( \begin{align*}
\text{same} & \}
\text{same} & \}
\end{align*} \)
Application: Collinear points.

Given n points, are there 3 (or more) collinear?

Solution. Apply duality. There are 3 collinear points iff the dual has 3 lines through a point. Construct the arrangement, and check for this. $O(n^2)$

Is there a faster algorithm? — see below
Levels in an arrangement

Any vertical line not through vertices orders the edges top to bottom.

Level $L_1$ = all edges that appear first (topmost) along such a vertical line: $L_1 \ L_2 \ L_5 \ L_7$
Level $L_i$ = all edges that appear in $i$-th place along such a line

Claim. Levels can be constructed in $O(n^2)$ time.
Claim. Levels can be constructed in $O(n^2)$ time.

- Compute arrangement $O(n^2)$
- Sort lines by slope — this gives edges in each $O(n \log n)$ level at $x = -\infty$
- Trace line $l$ through the arrangement

Total time $O(n^2)$. 
Levels in an arrangement

**Open problem:** what is the complexity of level $L_k$?
i.e. what is the worst case number of edges in level $L_k$?

Dual: given a set of points, how many subsets of size $k$ can be cut away with a line?
For $k = n/2$, how many *halving lines* can there be?

Best known bounds:

- $\Omega(n \log k)$  
  1973, Erdös et al., raised a bit by Toth, 2001

- $O(n k^{1/3})$  
  Tamal Dey, 1997

Also: find level $L_k$ (without constructing whole arrangement)
Application: Discrepancy problem.

Given n points in a unit square, do they provide a reasonable random sample?

discrepancy of half-plane $h = \mid$ area of square below $h$ – fraction of points below $h$ $\mid$

example:
7 points in shaded area; 13 points total
so fraction of points below $h$ is 7/13

Given n points, find the maximum discrepancy of any half-plane.

Arrangements give an $O(n^2)$ time algorithm for this.

nice presentation:  
Application: Discrepancy problem.

**Lemma.** Maximum discrepancy occurs

1. at line $h$ through 2 points, or

2. at line $h$ through 1 point and the point is the midpoint of the segment $h \cap \text{unit square}$

Proof.

- if $h$ goes through 0 points we can slide $h$ up or down to increase discrepancy.

- if $h$ goes through 1 point $p$

  increase discrepancy by rotating $h$ around $p$ unless $p$ is midpoint.
Solving the discrepancy problem via arrangements.

- Type 2 lines $h$ can be checked brute force
  \[ O(n^2) \text{ per point} \]
  \[ O(n^2) \text{ total} \]

- Type 1 points - $h$ thru 2 points
  use dual arrangement
  point $h^*$ through 2 lines
  at intersection of
  test all vertices of arrangement
  - the level gives # points above

Total \[ O(n^2) \] (area below $h$ takes $O(1)$)
Application: Ham sandwich theorem.

**Theorem.** Given a set of red points and a set of blue points in the plane, there exists a line that cuts both sets in half.

**General Ham-Sandwich Theorem**
(from ’30’s - 40’s).
In $\mathbb{R}^d$, any $d$ measurable objects can be cut in half by one $(d-1)$ dimensional hyperplane.

For discrete version in plane, there is an $O(n)$ time algorithm to find the halving line.

Can be viewed in terms of arrangements.
Application: Ham sandwich theorem.

**Theorem.** Given a set of red points and a set of blue points in the plane, there exists a line that cuts both sets in half.

Standard proof idea uses a rotating line.

In terms of duality and arrangements:

- The two $\frac{n}{2}$ levels intersect at a point.
- The halving line.
- The $n/2$ level of red lines.
- The point $p$ is solution to Ham sandwich.
3-SUM hardness

Can we test for 3 collinear points (for point set in the plane) faster than $O(n^2)$?

It is a “3-SUM-hard” problem, one of a large class of “equivalent” problems that all seem(ed) to need $O(n^2)$ time.

3-SUM problem: Given $n$ numbers, are there 3 that sum to 0? (repetition is allowed)

Exercise. Find an $O(n^2)$ time algorithm for 3-SUM.
[This is not too hard. Start by sorting the points.]

Lemma. If we could test for 3 collinear points in $o(n^2)$, then we could solve 3-SUM in $o(n^2)$.

Proof. Given $n$ numbers as input to 3-SUM, map each number $x$ to the point $(x,x^3)$.

Claim. 3 numbers $a,b,c$ sum to 0 iff the corresponding points are collinear.

Points collinear iff slopes $(a,a^3)$ to $(b,b^3) =$ slope $(b,b^3)$ to $(c,c^3)$

iff $\frac{b^3-a^3}{b-a} = \frac{c^3-b^3}{c-b}$

iff $b^2 + a^2 + ab = c^2 + b^2 + cb$

iff $b(a-c) = c^2 - a^2$ iff $b = \frac{c^2-a^2}{c-a}$

iff $a+c+b = 0$
recent breakthrough on 3-SUM:

an algorithm with run time $O(n^2 / (\log n / \log \log n)^{2/3})$


improved by Timothy Chan, 2018

very recent paper on “fine-grained” complexity lower bounds:

Hardness for Triangle Problems under Even More Believable Hypotheses: Reductions from Real APSP, Real 3SUM, and OV


The 3-SUM hypothesis, the APSP hypothesis and SETH are the three main hypotheses in fine-grained complexity. So far, within the area, the first two hypotheses have mainly been about integer inputs in the Word RAM model of computation. The "Real APSP" and "Real 3-SUM" hypotheses, which assert that the APSP and 3-SUM hypotheses hold for real-valued inputs in a reasonable version of the Real RAM model, are even more believable than their integer counterparts. Under the very believable hypothesis that at least one of the ...
Summary

- applications of arrangements

- testing collinearity and the 3-SUM problem.

References

- [CGAA] Chapter 8

- [Zurich notes] Chapter 8