Recall a problem we considered before: given n points, are there 3 (or more) collinear

By duality (points $\leftrightarrow$ lines) this becomes:
given n lines, do 3 of them intersect at a point.

To get an $O(n^2)$ algorithm, we study *line arrangements*.

A set of n lines in the plane partitions the plane into faces (cells), edges, vertices, called the *arrangement*. 
How many vertices, edges, faces for n lines?

A **degeneracy** is parallel lines or >2 lines through one point.

Exact bounds if there are no degeneracies (these decrease in case of degeneracy):

- Number of lines: \( n \)
- Number of vertices: \( \binom{n}{2} \) because every 2 lines intersect.
  
  - General case: \( \binom{n}{2} \)
  
  - General case: \( \binom{n}{2} \)
- Number of edges: each line crossed by \( n-1 \) others \( \implies n \) edges per line
  
  - General case: \( n^2 \)
- Number of faces: \( f_0 = 1 \) \( \implies f_n = n + (n-1) + (n-2) \ldots + 1 + f_0 = \binom{n+1}{2} + 1 \)
Constructing arrangements

input: n lines
output: list of faces, edges, vertices and all incidence relationships
(note: size is $\Theta(n^2)$)

Plane sweep would take $O(n^2 \log n)$ (because $n^2$ events, and log n per update)

There is an $O(n^2)$ time algorithm (deterministic, not randomized)

Idea: insert lines one by one
update each time.

Maintain cyclic order of edges around vertices
also keep order at circle at infinity
How to update after adding line $l_i$ to the line arrangement:

- Find intersection $x$ of $l_i$ and $l_j$.
- And edge $e$ on $l_i$ and a face $f$ containing $e$.
- Walk around $f$ to get next intersection with $l_i$.
- Hop to adjacent face $f_{i+1}$, continue until we return to $e$.
- Update all info about the arrangement as we go.

Time:
- Initialize $(\text{find } x, e, f)$ $O(n)$.
- Walking and updating: constant time per edge visited $O(\text{# edges in faces cut by } l_i)$.

We will show $O(i)$. 

2 kinds of updates:
- Hit $l_i$: switch faces incident to current edge.
- Hit vertex: use cyclic order to switch edges.
How to update after adding line $\ell_i$ to the line arrangement:

To bound the run time we need the Zone Theorem

Definitions. Let $A$ be an arrangement, and $\ell$ be a line not in $A$. The **zone** of $\ell$ in arrangement $A$ is $Z_A(\ell) = \{\text{faces of } A \text{ cut by } \ell\}$. The **size** of the zone is $z_A(\ell) = \sum \{\# \text{ edges in face } f : f \in Z_A(\ell)\}$. $z_n = \max\{z_A(\ell): \text{over all possible } \ell, A \text{ of } n \text{ lines}\}$

**Zone Theorem.** $z_n$ is $O(n)$. 
Algorithm ConstructArrangement \((L)\)

Input. Set \(L\) of \(n\) lines

Output. DCEL for \(A(L)\) in \(B(L)\)

1. Compute bounding box \(B(L)\)

2. Construct DCEL for subdivision induced by \(B(L)\)

3. for \(i = 1\) to \(n\) do

4. insert

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Example

Zone of \(l\) has 6 faces

\[ z(l) = 3 + 3 + 4 + 3 + 2 + 3 = 18 \]
Zone Theorem. \( z_n \) is \( O(n) \). (For non-degenerate case, \( z_n \leq 6n \).


Consequence: the incremental algorithm takes time \( O(n^2) \).

Proof
We will bound \( z_{A}(\ell) = \sum \{ \# \text{ edges in face } f : f \in Z_{A}(\ell) \} \)

Rotate so \( \ell \) is horizontal. Perturb so no other line is horizontal (this only increases the zone size).

Any face \( f \) in \( Z_{A}(\ell) \) has left and right boundary edges.

\[
z_{A}(\ell) = z(\ell) = z^{L}(\ell) + z^{R}(\ell)
\]

Claim. \( z^{L}(\ell) \leq 5n \) [\( \leq 3n \) for non-degenerate case]

This will prove the Zone Theorem.
Claim. \( z^l(l) \leq 5n \) [\( \leq 3n \) for non-degenerate case]

Proof

By induction on \( n \).

Basis

\[ \begin{array}{c}
\text{let } l_t \text{ has +slope} \\
\text{let } l_b \text{ has - slope}
\end{array} \]

\[ n=2 \]

\[ z^r(l) = 1 \]

\[ n=1 \]

\[ z^r(l) = 0 \]

\[ l \]

Let \( x \) = rightmost intersection with \( l \)

Find first intersection on \( l_i \) above/below

How may new left edges are created by adding \( l_i \)?

Claim: -1 left edge on \( l_i \) (none above \( l_t \) or below \( l_b \))

- left edge on \( l_t \) splits in 2
- left edge on \( l_b \) splits in 2

\[ \exists \text{ no other left edge is cut by } l_i \]

Increase in \( \# \) of left edges is 3. (non-degenerate case)

So \( \leq 3n \) left edges in non-degenerate case.
Claim. $z(l) \leq 5n \ [ \leq 3n \text{ for non-degenerate case}]$

Proof

Degeneracies allowed.
What happens in case there is degeneracy?

What if another line $l_j$ goes through $x$?
- still get left edge on $l_t$ split in two +1
- on $l_i$ get two new left edges +2
- on $l_j$ one left edge splits in two +1

So # left edges $\leq 5n$ +5

(if even more lines go through $x$, the bound goes down)
Arrangements in higher dimensions

For an arrangement of $n$ hyperplanes in $\mathbb{R}^d$

- the number of cells is $O(n^d)$

- Zone Theorem. The zone of a hyperplane has complexity $O(n^{d-1})$

In 3D, for $n$ planes, there are $O(n^3)$ cells, and a zone has complexity $O(n^2)$. 
Next lecture:

- applications of arrangements

- testing collinearity and the 3-SUM problem.
Summary

- arrangements

- size of parts of arrangements and the zone theorem

References

- [CGAA] Chapter 8

- [Zurich notes] Chapter 8