Curves of Bounded Width and the Asteroid Surveying Problem

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Abstract

We consider two generalizations to the well known puzzle "Sailor-in-the-Fog". First, we consider the natural generalization to three dimensions, in which, say, a disoriented diver is searching for the surface, which is in an unknown direction. In the video we review the "Sailor-in-the-Fog" problem and give an overview of the nature of the optimal solution, which is not as well known as the puzzle itself. Then we present a set of candidate solutions to the three dimensional problem, including a best-known-so-far curve of length 13.08 times the initial depth of the diver.

In the second part, we highlight the connection between curves of constant width and bounded width and the sailor problem. We consider the question of optimizing the length of given curves of constant and/or bounded width. We present a shortest path (or open curve) of width one and prove its optimality.

1 Introduction

A well known puzzle is the *Sailor-in-the-Fog* problem. In it, a fishing boat sets off to sea. While throwing the nets, a thick fog sets in, hiding the shore. The fishermen must now find a way back to harbor. The captain knows that they sailed straight out to sea for an hour at a steady pace. Then they came to a full stop and set anchor. He reckons that the shore is an hour sail away, in some unknown direction. The crew must now devise a path that will safely take the boat back to shore.

This problem was solved by Isbell in 1957 [Isb57], however, the optimal solution is not as well known as the statement of the problem. The proof of optimality is even less well known. We present a visual argument about the optimality of the solution by relying on the fact that shortest paths are preserved under certain mirror-image transformations.

The Sailor-in-the-Fog problem is set in two dimensions. Hence a natural question is to consider the generalization to three dimensions, in which starting from the origin, we wish to devise the shortest path that intersects all the planes at distance one. This problem was posed in the Open Problem session at the 14-th Canadian Conference on Computational Geometry in 2002 [DO02]. We term this puzzle the *night diver* problem.

2 Asteroid Surveying Problem

We give an interesting equivalent formulation, namely, the *asteroid surveying* problem. In this case, a spaceship starting from the surface of a spherical asteroid must survey the entire surface of the asteroid. The objective is to find the shortest path from which all points on the surface are visible.

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Interestingly, if we consider a polytope subscribing the sphere, then a Hamiltonian path of its vertices surveys the entire surface of the sphere. In general, the convex hull of a path which surveys the entire surface must contain the sphere.

We present a few candidate paths and give the shortest solution yet, which was found using a guided computer search. This solution has length equal to 6.04 times the diameter of the asteroid, plus the distance traversed from the surface to the starting point of the path. However, at this time the proof of its optimality remains an open question.

In the video, we omitted the portion of the path from the surface up for clarity of presentation. However, this length was taken into account when the proposed solution was computed.

3 Curves of Bounded Width

The "Sailor-in-the-Fog" problem is related to the concept of curves of constant width and bounded width. Indeed, any solution to the "Sailor-in-the-Fog" problem is a curve of width at least one, although the converse does not hold. Indeed, not every curve of width one is a solution to the "Sailor-in-the-fog" problem. Hence an interesting question is which is the shortest curve of width one. For the case of closed curves it is known that the solution is the set the so-called curves of constant width. These curves have the property that they can be used as "rollers" to smoothly roll an object on top of them. The circle is one well known such curve. Interestingly, there are infinitely many other such curves and even more surprisingly, all of them have perimeter π. This is a well known Theorem from the French mathematician Barbier.

If we do not restrict ourselves to closed curves then indeed shorter solutions are achievable. While all closed curves of constant width one have length π, the shortest open curve of width one has length 2.2782. To find the optimal solution we narrowed the set of potential candidates using auxiliary observations which are rather technical. All of these curves have start and ending point located on the x-axis. The family itself can be parameterized by the distance between the start point and end point. At this stage obtaining the optimal solution is a straightforward calculus optimization problem.

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