The Ultimate Strategy to Search on m Rays?*

Alejandro López-Ortiz† Sven Schuierer‡

Abstract

We consider the problem of searching on m current rays for a target of unknown location. If no upper bound on the distance to the target is known in advance, then the optimal competitive ratio is $1 + 2m^m/(m-1)^{m-1}$. We show that if an upper bound of D on the distance to the target is known in advance, then the competitive ratio of any search strategy is at least $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ which is again optimal—but in a stricter sense.

To show the optimality of our lower bound we construct a search strategy that achieves this ratio. Surprisingly, our strategy does not need to know an upper bound on the distance to the target in advance; it achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ if the target is found at distance D.

Finally, we also present an algorithm to compute the strategy that allows the robot to search the farthest for a given competitive ratio C.

1 Introduction

Searching for a target is an important and well studied problem in robotics. In many realistic situations the robot does not possess complete knowledge about its environment, for instance, the robot may not have a map of its surroundings, or the location of the target may be unknown [DI94, IK95, Kle92, LOS95, PY89].

Since the robot has to make decisions about the search based only on the part of its environment that it has explored before, the search of the robot can be viewed as an *on-line* problem. One way to judge the performance of an on-line search strategy is to compare the distance traveled by the robot to the length of the shortest path from its starting point s to the target t. The ratio of the distance traveled by the robot to the optimal distance from s to t over all possible locations of the target is called the *competitive ratio* of the search strategy [ST85].

*This research is supported by the DFG-Project "Diskrete Probleme", No. Ot 64/8-2.

†Faculty of Computer Science, University of New Brunswick, Fredericton, New Brunswick Canada, E3B 4A1, email: `alopez-o@unb.ca`

‡Institut für Informatik, Am Flughafen 17, Geb. 051, D-79110 Freiburg, Germany email: `schuiere@informatik.uni-freiburg.de`

We are interested in obtaining upper and lower bounds on the competitive ratio of searching on m concurrent rays. Here, a point robot is imagined to stand at the origin of m rays and one of the rays contains the target t whose distance to the origin is unknown. The robot can only detect t if it stands on top of it. It can be shown that an optimal strategy visits the rays in cyclic order and increases the step length each time by a factor of $m/(m-1)$ starting with a step length of 1 [BYCR93, Gal80]. The competitive ratio C_m achieved by this strategy is given by $1+2m^m/(m-1)^{m-1}$. If randomization is used, the optimal competitive ratio is given by the minimum of the function $1 + 2a^m/((a-1)\ln a)$, for $a > 1$ [Gal80, KRT97, KMSY94].

Searching on m rays has proven to be a very useful tool for searching in a number of classes of simple polygons, such as star-shaped polygons [LOS97], generalized streets [DI94, LOS96], HV-streets [DHS95], and θ-streets [DHS95, Hip94].

However, the proof of optimality for the above m-way ray searching strategy relies on the unboundedness of the rays, that is, on the fact that the target can be placed arbitrarily far away from the starting point of the rays [BYCR93, Gal80]. But, if we consider polygons and the robot is equipped with a range finder, then it is possible to obtain an upper bound D on the distance to the target. In this case it is implicitly assumed that the strategy for searching on m-rays remains optimal though no proof of this assumption has been presented yet [DHS95, DI94, LOS96].

In this paper we provide the first lower bound proof for searching on m bounded rays; more precisely, we investigate the question if the knowledge of an upper bound on the distance to the target provides an advantage to the robot.

Let C_m^D be the optimal competitive ratio to search on m rays where the distance to the target is at most D. As mentioned above it is assumed in the literature that C_m^D approaches C_m as D goes to infinity; yet, there is only a proof for the case $m = 2$ by López-Ortiz who shows that $9 - O(1/\log D)$ is a lower bound for the competitive ratio of searching on two rays [LO96]. Hipke *et al.* investigate the inverse problem, again for the case $m = 2$ [IKL97]. They consider the maximal reach of a strategy to search on the line if the competitive ratio of the strategy is given. The *reach* of a strategy X is the maximum distance D such that a target placed at a distance D to the origin is still detected by a robot using X if the competitive ratio of X equals C. Since C is given, a recurrence equation for the optimal reach can be derived. Using this recurrence equation Hipke *et al.* show that the maximal reach is continuous and strictly monotone in C [IKL97]. This in turn implies that C_2^D is strictly monotone in D and assumes all values in the interval $[3, 9]$.

In this paper we prove that

$$1 + 2m^m/(m-1)^{m-1} - O\left(1/\log^2 D\right) \tag{1}$$

is a lower bound on C_m^D, for general m; this also improves López-Ortiz' bound for $m = 2$. Moreover, we present a strategy that achieves a competitive ratio of the same form as Equation 1, albeit with a different constant factor in the "big-Oh" term. Here, D is the distance at which the target is discovered. Astonishingly, our strategy achieves this competitive ratio without knowing an upper bound on D in advance. These two results imply that the lower bound we present is asymptotically optimal.

Note that all previously proposed strategies have a competitive ratio of $1+2m^m/(m-1)^{m-1} - O(1/D)$ if the target is detected at distance D [BYCR93, Gal80]. Finally, we also present an algorithm to compute the maximal reach for a given competitive ratio C and arbitrary m—thus, generalizing the results by Hipke *et al.* [IKL97].

The paper is organized as follows. In the next section we give the basic definitions concerning searching on m rays. In Section 2 we show that an optimal strategy to search on m bounded rays visits the rays in a fixed cyclic order. We also derive a recurrence equation that is satisfied by an optimal strategy. In Section 3 we first consider searching on two rays to introduce our approach to analysing the competitive ratio of an optimal strategy. In Section 4 we generalize our ideas to the case of searching on m rays. Section 5 describes and analyses a strategy whose competitive ratio converges asymptotically as fast to $1+2m^m/(m-1)^{m-1}$ as the lower bound which we have shown before. Finally, in Section 6 we present an algorithm to compute the strategy with maximal reach for a given competitive ratio C.

2 Searching on m Bounded Rays

We are interested in the case that an upper bound D on the maximum distance of the target to the origin is known. Let X be a strategy to search on m bounded rays. Strategy X proceeds in *steps.* In each step the robot travels on one ray to a certain distance and, if it does not find the target, then it returns to the origin. Let x_i be the maximal distance to the origin and r_i the ray visited in Step i.

We define J_i as the index of the step in which ray r_i is visited the next time after Step i, that is, $J_i = \min\{j > i \mid r_j = r_i\}$. If there is no $j > i$ with $r_j = r_i$, then we define $J_i = i$. We represent X by the sequence of pairs (x_i, J_i).

Assume that the target is discovered in Step J_k in ray r. By the definition of J_k ray r was visited the last time before Step J_k in Step k and the distance d to the target is greater than x_k. The distance traveled by the robot to discover t is $d+2\sum_{i=0}^{J_k-1} x_i$. Since the target can be placed arbitrarily close to x_k by an adversary, the competitive ratio of Step k is given by

$$\sup_{d>x_k}(d+2\sum_{i=0}^{J_k-1} x_i)/d = \sup_{d>x_k} 1+2\sum_{i=0}^{J_k-1} x_i/d = 1+2\sum_{i=0}^{J_k-1} x_i/x_k.$$

The competitive ratio C_X of X is now given as the supremum of the competitive ratios over all steps.

The first step is a special case that we have not considered yet. If no information about the target is available, then one false move in the beginning may lead to an arbitrarily large competitive ratio. In order to avoid this problem we assume that a lower bound of one for the distance to the target t is known in advance that is, the target may be placed on any of the m rays somewhere in the interval $[1, D]$.

We denote the optimal competitive ratio of searching on m rays for a target that is placed at a distance of at most D from the origin by C_m^D. In the following we show that

$$C_m^D \geq 1 + 2\frac{m^m}{(m-1)^{m-1}} - O\left(\frac{1}{\log^2 D}\right).$$

2.1 Periodicity

In order to prove a lower bound on the competitive ratio, we first show that an optimal strategy—that is, a strategy with minimal competitive ratio—is *periodic* and *monotone*. In the following let $X = (x_0, \ldots, x_n)$ be a strategy to search on m bounded rays. Let r_k be the ray that the robot visits in Step k. Strategy X is *periodic* if $r_{k+m} = r_k$, for all $0 \leq k \leq n - m$. Strategy X is *monotone* if $x_k \leq x_{k+1}$, for all $0 \leq k \leq n - 1$.

Lemma 2.1 *If X is a strategy to search on m rays for a target that is placed at a distance of at most D from the origin, then there is a monotone strategy X^* with $C_{X^*} \leq C_X$.*

Proof: The proof is similar to the proof to the proof by Gal for the unbounded case [Gal80]. Let $X = (x_i)$ be a strategy to search m bounded rays and r_i the ray that is explored by X in the ith step. We define J_i as above.

Let $F_i(X) = \sum_{j=0}^{J_i-1} x_j/x_i$, for $0 \leq i \leq n$. If J_i does not equal i, then the competitive ratio in Step i of strategy X is given by $1 + 2F_i(X)$. If J_i equals i, that is, $x_i = D$ and Step i is the last step on ray r_i, then the competitive ratio in Step i of strategy X is bounded by

$$\frac{2\sum_{j=0}^{i-1} x_j + d}{d} \leq 1 + 2\frac{\sum_{j=0}^{i-1} x_j}{x_{J_i^{-1}}} = 1 + 2F_{J_i^{-1}}(X)$$

where J_i^{-1} is the index of the last visit of ray r_i before i and $d > x_{J_i^{-1}}$ is the distance from the origin to the target. Let I be the set of indices i with $J_i \neq i$. The competitive ratio C_X of X is now given by

$$C_X = \max_{i \in I} 1 + 2F_i(X).$$

If X is monotone, then there is nothing to show. So assume that there is a Step k, $0 \leq k \leq n - 1$ such that $x_{k+1} < x_k$. Let X^* be the search strategy which is equal to X except that for all steps $i \geq k$ the role of r_k and r_{k+1} is exchanged as are x_k and x_{k+1}. This can be achieved by setting $(x_k^*, J_k^*) = (x_{k+1}, J_{k+1})$ and $(x_{k+1}^*, J_{k+1}^*) = (x_k, J_k)$. For all other Steps i, $(x_i^*, J_i^*) = (x_i, J_i)$ unless $x_{k+1}^* = D$, in which case we set $J_{k+1}^* = k + 1$ (and not equal to k as implied by the rule above). Note that $x_k^* = x_{k+1} = D$ is not possible since $x_{k+1} < x_k \leq D$. Let I^* be the set of indices i with $J_i^* \neq i$. We want to show that $C_{X^*} = \max_{i \in I^*} 1 + 2F_i(X^*) \leq \max_{i \in I} 1 + 2F_i(X) = C_X$. Obviously, $F_i(X)$ and $F_i(X^*)$ differ only for the indices J_k^{-1}, J_{k+1}^{-1}, k, $k+1$ which we are going to consider more closely in the following.

First we assume that Step k is not the last step on ray r_k. (As mentioned before, Step $k+1$ is never the last step on ray r_{k+1} as $x_{k+1} < x_k \le D$.)

$$\begin{aligned} F_k(X) &= \frac{\sum_{i=0}^{J_k-1} x_i}{x_k} = \frac{\sum_{i=0}^{J^*_{k+1}-1} x^*_i}{x^*_{k+1}} = F_{k+1}(X^*) \qquad \text{and} \\ F_{k+1}(X) &= \frac{\sum_{i=0}^{J_{k+1}-1} x_i}{x_{k+1}} = \frac{\sum_{i=0}^{J^*_k-1} x^*_i}{x^*_k} = F_k(X^*). \end{aligned}$$

Here the equalities follow from the fact that $J^*_{k+1} = J_k \ge k+2$ and $J^*_k = J_{k+1} \ge k+2$, that is, the exchange of x_k and x_{k+1} does not play a role in the summation. Next we consider Steps J^{-1}_{k+1} and J^{-1}_k. Note that $J^{-1*}_k = J^{-1}_k$ and $J^{-1}_{k+1}{}^* = J^{-1}_{k+1}$. Moreover,$J_{J^{-1}_k} - 1 = J^*_{J^{-1*}_k} - 1 = k-1$; therefore, $F_{J^{-1}_k}(X) = F_{J^{-1*}_k}(X^*)$. This leaves us with Step J^{-1}_{k+1}. We have

$$F_{J^{-1}_{k+1}}(X) = \frac{\sum_{i=0}^{k} x_i}{x_{J^{-1}_{k+1}}} \ge \frac{\sum_{i=0}^{k} x_i - x_k + x_{k+1}}{x_{J^{-1}_{k+1}}} = \frac{\sum_{i=0}^{k-1} x^*_i + x^*_k}{x^*_{J^{-1}_{k+1}{}^*}} = F_{J^{-1}_{k+1}{}^*}(X^*).$$

Now assume that Step k is the last step on ray r_k and $D = x_k > x_{k+1}$. Then, $F_{k+1}(X^*) \le F_{J^{-1}_{k+1}{}^*}(X^*)$. As above we obtain $F_k(X^*) = F_{k+1}(X)$, $F_{J^{-1}_k}(X^*) = F_{J^{-1}_k}(X)$ and $F_{J^{-1}_{k+1}{}^*}(X^*) \le F_{J^{-1}_{k+1}}(X)$. Hence, the competitive ratio of Strategy X^* is no more than the competitive ratio of strategy X.

By performing bubble-sort on strategy X we see that there is a monotone strategy that has a competitive ratio no more than X which proves the claim. □

By Lemma 2.1 it suffices to consider monotone strategies in the following. Note that if X is monotone, then the last m steps of X all have length D, that is, there is an optimal strategy with $x_{n-m+1} = \cdots x_n = D$ and the set of indices i with $J_i \ne i$ equals $\{0, \ldots, n-m\}$.

Lemma 2.2 *If X is a strategy to search on m rays for a target that is placed at a distance of at most D from the origin, then there is a periodic strategy X^* with $C_{X^*} \le C_X$.*

Proof: Let X be strategy to search on m bounded rays. By Lemma 2.1 we can assume that X is monotone. We follow the proof idea of Yin [Yin94]. Let X^* consist of the same sequence of numbers except that X^* is now considered a periodic strategy. We consider the competitive ratios C_k of X and C^*_k of X^* in Step k. It suffices to show that, for every $0 \le k \le n-m$, there is a $0 \le j \le n-m$ with $C^*_k \le C_j$. As mentioned above we do not need to consider the indices $n-m+1 \le k \le n$. So consider

$$C^*_k = 1 + 2\frac{\sum_{i=0}^{k+m-1} x_i}{x_k},$$

for some $0 \le k \le n-m$. For each ray r_j, $1 \le j \le m$, let k_j be the first time X explores ray r_j after Step k. Since $x_j < D$, for all $0 \le j \le n-m$, k_j exists, for all

$1 \le j \le n-m$. Note that there is one ray r_l such that $k_l \ge k+m$. If r_l is explored before Step k, then let $j_l \le k$ be the index of the last exploration; otherwise let $j_l = -1$ and $x_{j_l} = 1$. In both cases $x_{j_l} \le x_k$ since X is monotone and

$$C_k^* = 1 + 2\frac{\sum_{i=0}^{k+m-1} x_i}{x_k} \le 1 + 2\frac{\sum_{i=0}^{k_l-1} x_i}{x_{j_l}} = C_{j_l},$$

which implies that the competitive ratio of X is at least as large as the competitive ratio of X^*. $\square$

2.2 A Recurrence Equation

In the following we assume that X is an optimal periodic, monotone strategy. As mentioned before F_k simplifies in this case to $F_k(X) = \sum_{i=0}^{k+m-1} x_i/x_k$, for $k = 0, \ldots, n-m$ and $C_X = \max_{0 \le i \le n-m} 1 + 2F_i(X)$. We now show that the values x_i satisfy a recurrence equation. The following lemma was proven by Katsoupias and Papadimitriou for the special case $m = 2$ with unbounded rays [KPY96].

Lemma 2.3 *If X^* is an optimal strategy, then $1 + 2F_k(X^*) = C_m^D$, for all $0 \le k \le n-m$.*

Proof: The proof is by contradiction. It is based on the observation that F_k is the only function which is decreasing in x_k and all other functions F_i with $i \ge k-m+1$ are increasing in x_k [KPY96]. So if there is an index k with $1 + 2F_k(X) < C_m^D$, then there is an $\varepsilon > 0$ and a $\delta > 0$ such that if x_k is decreased by ε, then $1 + 2F_k(X') \le C_m^D - \delta$ if X' is the sequence where x_k is replaced by $x_k - \varepsilon$ and, in addition, $1 + 2F_i(X') \le C_m^D - \delta$, for all $k - m + 1 \le i \ne k \le n - m$.

Let X be a sequence with competitive ratio C_m^D and l_X the minimal index for X with $1 + 2F_k(X) < C_m^D$. Let X^* be a sequence with competitive ratio C_m^D such that $l^* = l_{X^*}$ is minimal among all such sequences. If $l^* \ge m-1$, then we can apply the above argument and obtain a sequence X' from X^* with $1 + 2F_k(X') < C_m^D - \delta$, for all $l^* - m + 1 \le k \le n-m$—in contradiction to the minimality of l^*. If $l^* < m-1$, then we can apply the above argument and obtain a sequence X' from X^* with $1 + 2F_k(X') < C_m^D - \delta$, for all $0 \le k \le n-m$—in contradiction to the minimality of C_m^D. Hence, there is no sequence X with competitive ratio C_m^D and an index k with $1 + 2F_k(X) < C_m^D$. $\square$

In the following let $c_m^D = (C_m^D - 1)/2$. Lemma 2.3 implies that the step lengths x_i of an optimal strategy X satisfy the following recurrence equation.

$$\frac{\sum_{i=0}^{k+m-1} x_i}{x_k} = c_m^D \quad \text{or} \quad \sum_{i=0}^{k+m-1} x_i = c_m^D x_k, \tag{2}$$

for $0 \le k \le n-m$. An additional constraint is given by the first time the m-th ray is visited; here, the competitive ratio is given by

$$1 + 2\sum_{i=0}^{m-2} x_i \le 1 + 2c_m^D \tag{3}$$

as in steps $0, \ldots, m-2$ the first $m-1$ rays are explored. If we multiply c_m^D by a factor of x_{-1} where $0 < x_{-1} \leq 1$, then we achieve equality in (3) and we can view (3) as a special case of (2) for $k = -1$. Hence, we assume in the following that Equation 2 holds for all $-1 \leq k \leq n-m$.

The linear equation system (2) consists of $n - m + 2$ linearly independent equations for the $n+1$ step lengths $(x_{-1}, x_0, x_1, \ldots, x_{n-1})$ of X (x_n is irrelevant since x_n does not appear in Equation 2). Since we are given the values of $x_{n-m+1} = \cdots = x_{n-1} = D$, the $n+1$ solutions $(x_{-1}, x_0, x_1, \ldots, x_{n-1})$ are uniquely defined once we are given c_m^D, D, and n. We are interested in the question how large c_m^D has to be for a given D such that there is an n and a *positive* solution $(x_{-1}, x_0, x_1, \ldots, x_{n-1})$ with $x_{-1} \leq 1$. As this question seems to be rather difficult to answer, we transform Equation 2 into a simpler form.

Lemma 2.4 *The values x_i satisfy the following recurrence equation*

$$x_{k+m-1} - c_m^D x_k + c_m^D x_{k-1} = 0, \tag{4}$$

for $0 \leq k \leq n-m$.

Proof: By Equation 2 we have

$$\sum_{i=0}^{k+m-1} x_i = c_m^D x_k,$$

for $0 \leq k \leq n-m$. The same equation also holds for $k-1$. Hence,

$$\sum_{i=0}^{k+m-1} x_i = c_m^D x_k \qquad \text{and} \qquad \sum_{i=0}^{k+m-2} x_i = c_m^D x_{k-1}.$$

By subtracting the second equation from the first we obtain Equation 4, for $0 \leq k \leq n-m$ as claimed. □

Unfortunately, we obtain only $n-m+1$ equations in this way—one too few—and the sequence X is not completely determined anymore by Equation 4 and the $m-1$ initial values $x_{n-m+1} = \cdots = x_{n-1} = D$. One option to get around this problem is to add the first or last equation of (2) as an additional constraint to recurrence equation (4). However, as this destroys the uniformity of the recurrence equation (4), we take a different approach and introduce one more initial value.

We reduce the m values $x_{n-m}, x_{n-m+1}, \ldots, x_{n-1}$ to the value $D^* = c_m^D/(c_m^D - 1)x_{n-m-1}$. The new sequence X' we obtain in this way—that is, $x'_i = x_i$, for $0 \leq i \leq n-m-1$, and $x_{n-m} = \cdots = x_{n-1} = D^*$—does not fulfill Equation 4 anymore but only

$$x'_{k+m-1} - c_m^D x'_k + c_m^D x'_{k-1} \quad \leq \quad 0, \qquad \text{that is,} \qquad \frac{x'_{k+m-1}}{x'_k - x'_{k-1}} \quad \leq \quad c_m^D$$

for all $0 \leq k \leq n-m$, and, in addition, by our choice of D^*

$$x'_{n-1} - c_m^D x'_{n-m} + c_m^D x'_{n-m-1} = 0. \tag{5}$$

It is easy to see that $x_{n-m-1} \geq (1-\alpha_m)x_{n-m}$ where $\alpha_m = (c_m^D - 1)/(c_m^D(m-1))$ and $x_{n-m} \geq D/(2c_m^D/m) \geq D/(2e)$; here, we make use of the fact that $c_m^D \leq m^m/(m-1)^{m-1} \leq me$. Hence, $D^* \geq (m-2)/(2e(m-1))D$. Of course, $\sum_{i=0}^{m-2} x'_i \leq c_m^D x'_{-1} \leq c_m^D$ still holds for X'.

Similar to the proof of Lemma 2.3 we can now show that there is a minimal value $c^* \leq c_m^D$ and a sequence X^* that satisfies $x^*_{n-m} = \cdots = x^*_{n-1} = D^*$ as initial conditions, $\sum_{i=0}^{m-2} x^*_i \leq c^* x^*_{-1} \leq c^*$ and, for all $0 \leq k \leq n-m$,

$$x^*_{n-1} - c_m^D x^*_{n-m} + c_m^D x^*_{n-m-1} = 0. \tag{6}$$

If we were given the m values $x^*_{-1}, x^*_0, \ldots, x^*_{m-2}$ (which we do not know), then the sequence $(x^*_{-1}, x^*_0, x^*_1, \ldots, x^*_{n-1})$ would be completely determined by Equation 6, D^*, and c^*; however, we *do* know the m values of $x^*_{n-m}, \ldots, x^*_{n-1}$. In order to make use of this information we consider the sequence Y of the values of X^* in reverse order, that is, $y_i = x^*_{n-i-1}$, for $i = 0, \ldots, n$. The sequence Y satisfies the recurrence equation

$$y_{k+m} - y_{k+m-1} + \frac{1}{c^*} y_k = 0, \tag{7}$$

for all $0 \leq k \leq n-m$.

In the following let $Y_{c,D}$ be the *infinite* sequence that is given by Equation 7 (with $c^* = c$) and the initial values $y_0 = y_1 = \cdots = y_{m-1} = D$. $Y_{c,D}$ is completely determined by Equation 7 and $y_0, \ldots, y_{m-1}$. The sequence Y is a positive prefix of Y_{c^*,D^*}. Note that $Y_{c,D}$ may contain negative elements for some $k > n$ if $c < m^m/(m-1)^{m-1}$. We will show the following lemma.

Lemma 2.5 *If $c < m^m/(m-1)^{m-1} - O(1/\log^2 D)$, then there is an index $k \geq m$ for the sequence $Y_{c,D} = (y_0, y_1, \ldots)$ with $y_{k-m} > c^2$ and $y_k < 0$.*

Note that constant in the "big-Oh" term above depends on m. In the proof of Lemma 2.5 we will present an upper bound on the constant.

Assuming we have shown Lemma 2.5, we can easily prove that the competitive ratio of any strategy to search on m rays in the interval $[1, D]$ is bounded from below by $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$.

Theorem 2.6 *If $c < m^m/(m-1)^{m-1} - O(1/\log^2 D)$, then there is no strategy X with a competitive ratio of $1+2c$ that searches on m rays for a target of distance at most D to the origin.*

Proof: The proof is by contradiction. Assume there is a strategy X with a competitive ratio of $1+2c$ that searches on m rays for a target of distance at most D to the origin. This implies that $c \geq c_m^D$.

Let X be an optimal strategy to search on m rays. By Lemma 2.2 and the above considerations we can assume that X is periodic and satisfies Equation 2.

As above we construct a sequence X^* that satisfies Equation 6, for some $c^* \leq c_m^D$, $x^*_{n-m} = \cdots = x^*_{n-1} = D^*$, with $D \geq D^* \geq (m-2)D/((m-1)2e)$ and $\sum_{i=0}^{m-2} x_i^* \leq c^* x^*_{-1} \leq c^*$. As can be easily seen, the values x_i^* also satisfy $\sum_{i=0}^{m-1} x_i^* \leq c^* x_0^*$.

We define the sequence $Y = (y_0, \ldots, y_n)$ by $y_i = x^*_{n-i-1}$, for $0 \leq i \leq n$, where n is the length of X^*. The sequence $(y_0, \ldots, y_n)$ is a positive prefix of the infinite sequence Y_{c^*,D^*}. Since $c^* \leq c < m^m/(m-1)^{m-1} - O(1/\log^2 D) = m^m/(m-1)^{m-1} - O(1/\log^2 D^*)$, Lemma 2.5 implies that there is an index k for Y_{c^*,D^*} with $y_{k-m} \geq c^{*2}$ and $y_k < 0$. Since $y_k < 0$, n is at most k. Since $y_{n-1} \leq \sum_{i=n-m+1}^{n-1} y_i \leq c^*$ and $y_{n-m} \geq y_{k-m} \geq c^{*2}$, we have $\sum_{i=n-m}^{n-1} y_i > y_{n-m} \geq c^{*2} \geq c^* y_{n-1}$—a contradiction.□

2.3 The Characteristic Equation

In the following we are only concerned with proving Lemma 2.5. The recurrence equation for $Y_{c,D}$ has the characteristic equation

$$\lambda^m - \lambda^{m-1} + \frac{1}{c} = 0 \qquad \text{or} \qquad c = \frac{1}{\lambda^{m-1}(1-\lambda)}. \tag{8}$$

We first note that since $\lambda^{m-1}(1-\lambda) < 0$, for $\lambda > 1$, there is no positive real root larger than one. On the other hand, if there is a positive real root λ of Equation 8 with $\lambda < 1$, then $c \geq \inf_{1>\lambda>0} 1/(\lambda^{m-1}(1-\lambda)) = m^m/(m-1)^{m-1}$ and we are done. Hence, we can assume in the following that there is no positive real root of Equation 8 and we only need to investigate the complex and negative roots of Equation 8 in more detail.

3 Solving the Recurrence Equation for $m = 2$

In order to illustrate our approach we present the case $m = 2$ in greater detail. We can assume that c is less than $m^m/(m-1)^{m-1} = 4$ in the following.

3.1 An Explicit Solution

For $m = 2$ Equation 8 reduces to

$$\lambda^2 - \lambda + 1/c = 0 \tag{9}$$

with the solutions

$$\lambda \;=\; \frac{1}{2}\left(1 + i\sqrt{\frac{4-c}{c}}\right) \qquad \text{and} \qquad \overline{\lambda} \;=\; \frac{1}{2}\left(1 - i\sqrt{\frac{4-c}{c}}\right).$$

Here, $\overline{\lambda}$ denotes the conjugate of λ. Hence, the solution of Equation 7 in the case $m = 2$ is given by

$$y_k = a\lambda^k + \overline{a}\overline{\lambda}^k = 2Re(a\lambda^k) \tag{10}$$

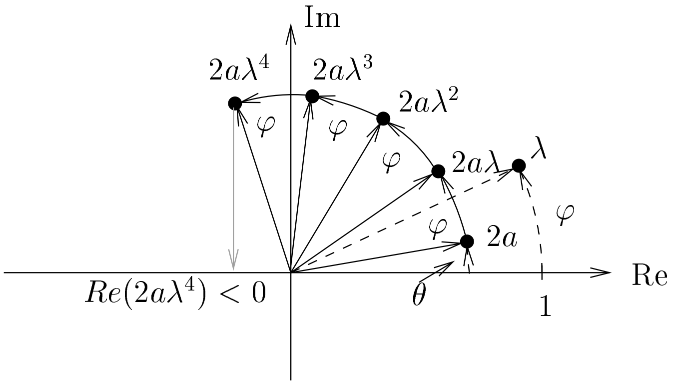

Figure 1: The sequence $2a\lambda^k$ turns by an angle of φ towards the second quadrant with each iteration. (For simplicity, we assume $|\lambda| = 1$.)

where Re denotes the real part of a complex number. The coefficients a and $\overline{a}$ are the solutions of the equation system

$$\begin{array}{ccccc} a & + & \overline{a} & = & y_0 = D \\ a\lambda & + & \overline{a}\overline{\lambda} & = & y_1 = D \end{array}$$

which solves to

$$a \;=\; \frac{D}{2}\left(1 - i\sqrt{\frac{c}{4-c}}\right) \qquad \text{and} \qquad \overline{a} = \frac{D}{2}\left(1 + i\sqrt{\frac{c}{4-c}}\right).$$

3.2 Polar Coordinates

If we consider the polar-coordinates of λ and $\overline{\lambda}$, that is, we set $\lambda = \rho e^{i\varphi}$ and $\overline{\lambda} = \rho e^{i(-\varphi)}$, then $\rho = \sqrt{1/c}$ and $\varphi = \arctan(\sqrt{(4-c)/c})$. Similarly, if $a = \sigma e^{i\theta}$ and $\overline{a} = \sigma e^{i(-\theta)}$, then $\sigma = D/\sqrt{4-c}$ and $\theta = -\arctan(\sqrt{c/(4-c)})$. The step length y_k is now given by

$$y_k = a\lambda^k + \overline{a}\overline{\lambda}^k \;=\; \frac{2D}{\sqrt{c^k(4-c)}}\cos\left(k\varphi + \theta\right). \tag{11}$$

If we visualize the above equation in the complex plane, then y_k is the projection of the vector of $2a\lambda^k$ onto the x-axis by Equation 10. Since by multiplying two complex numbers their polar angles are added, the sequence $2a\lambda^k$ turns by an angle of φ towards the second quadrant with each iteration. Once $2a\lambda^k$ is in the second quadrant, $2Re(a\lambda^k)$ is negative. This is illustrated in Figure 1 (see also [Hip94, IKL97, Kle97]).

We show that D can be chosen large enough such that there is an index n_0 with $y_{n_0} < 0$ and $y_{n_0-2} > c^2$ which proves Lemma 2.5. Of course, we are interested in the smallest D for which the above inequalities holds. Let n_0 be the first index such that $y_{n_0} < 0$, that is,

$$\cos(n_0\varphi + \theta) < 0 \qquad \text{or} \qquad n_0 = \left\lceil \frac{\pi/2 - \theta}{\varphi} \right\rceil = \left\lceil \frac{\pi/2 + \arctan\left(\sqrt{\frac{c}{4-c}}\right)}{\arctan\left(\sqrt{\frac{4-c}{c}}\right)} \right\rceil.$$

Since n_0 is the smallest k such that $y_k < 0$,

$$(n_0 - 2)\varphi + \theta \leq \frac{\pi}{2} - \varphi. \tag{12}$$

W.l.o.g. we assume that y_{n_0} belongs to ray r_1. Since the search alternates between the two rays, the last point visited on ray r_1 has a distance of

$$y_{n_0-2} \overset{(11,12)}{\geq} \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \cos\left(\frac{\pi}{2} - \varphi\right) = \frac{D}{\sqrt{c^{n_0-2}}} \tag{13}$$

to the origin.

We first consider the case that $c \in [1, 3)$. In this case $n_0 \leq \pi/\arctan(1/3) = 6$ and

$$y_{n_0-2} \geq \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \frac{\sqrt{4-c}}{2} \geq \frac{D}{\sqrt{c^4}} \geq \frac{D}{9}.$$

If $D > 81$, then $y_{n_0-2} > 9 > c^2$ and $y_{n_0} < 0$ which proves Lemma 2.5 for $m = 2$ and $c < 3$.

Now assume that $c \in [3, 4]$. Then, we have

$$n_0 = \left\lceil \frac{\arctan\left(\sqrt{\frac{c}{4-c}}\right) + \frac{\pi}{2}}{\arctan\left(\sqrt{\frac{4-c}{c}}\right)} \right\rceil \leq \frac{\pi/2 + \pi/2}{3/4\sqrt{(4-c)/c}} \leq \frac{4\pi}{3}\sqrt{\frac{c}{4-c}} \leq \frac{9}{\sqrt{4-c}}. \tag{14}$$

The first inequality stems from the fact that

1. $c \geq 3$, that is, $\sqrt{(4-c)/c} \leq 1/\sqrt{3}$ and
2. $\arctan(x)' = 1/(1+x^2)$, that is, $\arctan(x) \geq x/(1+x^2)$ since arcus tangens is concave on the positive axis. Hence, $\arctan(\sqrt{(4-c)/c}) \geq \sqrt{(4-c)/c}/(1 + \sqrt{1/3}^2)$.

We obtain $y_{n_0-2} \geq D/\sqrt{c^{n_0-2}} \overset{(14)}{\geq} D/\sqrt{c^{9/\sqrt{4-c}}}$.

Lemma 3.1 *If $3 \leq c < 4 - 81/\log^2(D/16)$, then $D/\sqrt{c^{9/\sqrt{4-c}}} > c^2$.*

Proof: We have

$$\begin{aligned} c &< 4 - \frac{81}{\log^2(D/16)} \overset{(\log c<2)}{\Rightarrow} & \log D &> \left(\frac{4.5}{\sqrt{4-c}} + 2\right)\log c \Rightarrow \\ D^2 &> c^{9/\sqrt{4-c}+4} \quad \Rightarrow & \frac{D}{\sqrt{c^{9/\sqrt{4-c}}}} &> c^2. \end{aligned}$$

□

Let $3 \leq c < 4 - 81/\log^2(D/16)$. Lemma 3.1 implies that $y_{n_0-2} > c^2$ and $y_{n_0} < 0$ which proves Lemma 2.5 for $m = 2$ and $c \geq 3$.

4 Solving the Recurrence Equation for the General Case

We now return to the general case. As for the case $m = 2$ we want to show that if there are only complex or negative solutions to Equation 8, then the contribution of a solution becomes negative after a sufficiently large number of steps. However, the details are much more complicated than in the case $m = 2$ since we have many roots of Equation 8 and the solutions cannot be computed explicitly. In order to get around this problem we use estimates on the angles and radii of the polar coordinates of the roots.

In the following we show that there is one root λ which has the largest radius among all roots of Equation 8. After a sufficiently large number of steps the contribution of λ dominates the contribution of all other roots. Once the contribution of λ becomes negative in Step k so does the step length y_k. This limits the number of steps Y. Since D can grow at most exponentially in the number of steps of Y, we also obtain a bound on D in this way.

Let $\lambda_0, \ldots, \lambda_{m-1}$ be the roots of Equation 8. The solution of the recurrence is given by

$$y_k = a_0\lambda_0^k + a_1\lambda_1^k + \cdots + a_{m-1}\lambda_{m-1}^k.$$

We first investigate the structure of the roots λ_i, $0 \le i \le m-1$. Let λ be a complex root of Equation 8. We consider the polar coordinates of λ, that is, we set $\lambda = \rho e^{i\varphi}$. We start off with a simple observation about the relationship between the radius and the polar angle of a root.

Lemma 4.1 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 8, then $\rho = \sin(m-1)\varphi/\sin m\varphi$ and $1/c = \rho^{m-1}(\sin\varphi/\sin m\varphi)$.*

Proof: Let $\lambda = \rho e^{i\varphi}$ be a complex root of Equation 8. We have $\lambda^{m-1} = \rho^{m-1}e^{i(m-1)\varphi}$ and

$$\begin{aligned}\lambda^{m-1}(\lambda-1) &= \rho^{m-1}\left(\cos(m-1)\varphi + i\sin(m-1)\varphi\right)\left(\rho\cos\varphi - 1 + i\rho\sin\varphi\right)\\ &= \rho^{m-1}\left(\rho\cos m\varphi - \cos(m-1)\varphi + i(\rho\sin m\varphi - \sin(m-1)\varphi)\right).\end{aligned}$$

Since $\lambda^{m-1}(\lambda-1) = -1/c \in I\!R$, we obtain

$$\rho\sin m\varphi - \sin(m-1)\varphi = 0 \qquad \text{or} \qquad \rho = \frac{\sin(m-1)\varphi}{\sin m\varphi}. \tag{15}$$

The second claim follows from the equalities

$$\begin{aligned}1/c &= -\lambda^{m-1}(\lambda-1) = \rho^{m-1}(\cos(m-1)\varphi - \rho\cos m\varphi)\\ &= \rho^{m-1}\left(\cos(m-1)\varphi - \frac{\sin(m-1)\varphi}{\sin m\varphi}\cos m\varphi\right) = \rho^{m-1}\frac{\sin\varphi}{\sin m\varphi}.\end{aligned}$$

□

Lemma 4.1 has the following consequence.

Corollary 4.2 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 8, then λ is solely determined by φ.*

4.1 The Polar Angle of a Root

We first concentrate on the polar angle of a root λ of Equation 8.

Lemma 4.3 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 8 and $0 \le \varphi \le \pi$, then $\varphi \in [2k\pi/(m-1), (2k+1)\pi/m]$, for some $0 \le k \le \lfloor m/2 \rfloor - 1$.*

Proof: Let $\lambda = \rho e^{i\varphi}$ be a complex root of Equation 8. Equation 15 implies that since $\rho > 0$ both $\sin m\varphi$ and $\sin(m-1)\varphi$ have the same sign, that is, $m\varphi$ and $(m-1)\varphi$ either both belong to $[2k\pi, (2k+1)\pi]$ or to $[(2k+1)\pi, (2k+2)\pi]$. Since $1/c = \rho^{m-1} \sin\varphi / \sin m\varphi > 0$ and $\rho > 0$ as well as $\sin\varphi \ge 0$ (since $0 \le \varphi \le \pi$), a second condition is $\sin m\varphi > 0$ which implies $\varphi \in [2k\pi/(m-1), (2k+1)\pi/m]$, for some $0 \le k \le \lfloor m/2 \rfloor - 1$ as claimed. □

In fact, each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ contains one root of Equation 8.

Lemma 4.4 *For $0 \le k \le \lfloor m/2 \rfloor - 1$, there is exactly one root $\lambda_k = \rho_k e^{i\varphi_k}$ of Equation 8 with $\varphi_k \in [2k\pi/(m-1), (2k+1)\pi/m]$.*

Proof: Since λ is a continuous function of φ by Lemma 4.1, it suffices to show that $1/(\lambda^{m-1}(1-\lambda))$ is monotone in φ and that $1/(\lambda^{m-1}(1-\lambda))$ assumes a value less than and greater than c, for each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ with $0 \le k \le \lfloor m/2 \rfloor - 1$.

Monotonicity follows immediately from considering the derivative of $1/(\lambda^{m-1}(1-\lambda))$ with respect to φ.

Hence, there is at most one root of Equation 8 for every interval $[2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \le k \le \lfloor m/2 \rfloor - 1$. Since $\sin\varphi/\sin m\varphi$ is continuous over $[2k\pi/(m-1), (2k+1)\pi/m]$ and its values range from ∞ to 0, there is also at least one root of of Equation 8 with a polar angle in $[2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \le k \le \lfloor m/2 \rfloor - 1$. □

The above roots account for $\lfloor m/2 \rfloor$ roots of Equation 8. If m is odd, then there is one root $\lambda_{\lfloor m/2 \rfloor}$ with $\varphi_{\lfloor m/2 \rfloor} = 2\lfloor m/2 \rfloor \pi/(m-1) = (2\lfloor m/2 \rfloor + 1)\pi/m = \pi$, that is, $\lambda_{\lfloor m/2 \rfloor}$ is a negative real root. It is easy to see that the remaining $\lfloor m/2 \rfloor$ roots are given by the conjugates $\overline{\lambda}_k = \rho_k e^{-i\varphi_k}$ of λ_k as in the case $m = 2$.

Let φ_k be the angle of the root in $[2k\pi/(m-1), (2k+1)\pi/m]$. In the following we calculate a lower bound on the size of φ_0 if $c < m^m/(m-1)^{m-1}$.

Lemma 4.5

$$\varphi_0 \ge \min\left\{ \frac{1}{m^{3/2}} \sqrt{\frac{m^m}{(m-1)^{m-1}} - c}, \frac{1}{\sqrt{3}m} \right\}.$$

Proof: We assume that $\varphi_0 \in [0, \pi/\sqrt{3}m]$ since if $\varphi_0 \geq \pi/\sqrt{3}m$, then the claim trivially holds.

$$
\begin{aligned}
c = \frac{\sin m\varphi_0}{\rho_0^{m-1}\sin\varphi_0} &= \left(\frac{\sin m\varphi_0}{\sin(m-1)\varphi_0}\right)^{m-1}\frac{\sin m\varphi_0}{\sin\varphi_0} \\
&\geq \left(1+\frac{(m-m^3\varphi_0^2/6)\varphi_0}{(m-1)\varphi_0}\right)^{m-1}\frac{m\varphi_0-(m\varphi_0)^3/6}{\varphi_0} \\
(\varphi_0 \leq \pi/\sqrt{3}m) &\geq \left(1+\frac{m-m^3\pi^2/(18m^2)}{(m-1)}\right)^{m-1}\frac{m\varphi_0-(m\varphi_0)^3/6}{\varphi_0} \\
&\geq \left(\frac{m}{m-1}\right)^{m-1}\left(m-\frac{m^3\varphi_0^2}{6}\right) \geq \left(1-\frac{m^2\varphi_0^2}{6}\right)\frac{m^m}{(m-1)^{m-1}}.
\end{aligned}
$$

Here we use that by the Taylor-expansion of $\sin x - x^3/6 \leq \sin(x) \leq x$ if $x \geq 0$. Since $m^m/(m-1)^{m-1} < em$, we have

$$\varphi_0 \geq \sqrt{\frac{6(m^m/(m-1)^{m-1}-c)}{em^3}} \geq \frac{1}{m^{3/2}}\sqrt{\frac{m^m}{(m-1)^{m-1}}-c}$$

as claimed. □

4.2 The Radius of a Root

We now consider the radius of a root of Equation 8. Let ρ_k be the radius of λ_k. In the following we show that $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{\lceil m/2\rceil-1}$.

Lemma 4.6 *For all* $0 \leq k \leq \lceil m/2\rceil - 2$, $\rho_k \geq \rho_{k+1}$.

Proof: For $0 \leq k \leq \lceil m/2\rceil - 1$, let f_φ be the function

$$f_\varphi(\rho) = |\lambda^{m-1}(1-\lambda)| = \rho^{m-1}\sqrt{\rho^2 - 2\rho\cos\varphi + 1}.$$

We show that $f_{\varphi_k}(\rho)$ is monotonely increasing in ρ, for $1 \leq k \leq \lceil m/2\rceil - 1$. If we consider the derivative of f_{φ_k} with respect to ρ, then it is easy to see that f_{φ_k} can only have an extremum if

$$\sin\varphi_k \leq \frac{1}{2m-1} \qquad \Rightarrow \qquad \varphi_k \leq \arcsin\left(\frac{1}{2m-1}\right) \leq \frac{2}{2m-1} < \frac{1}{m-1},$$

since $m \geq 3$ and $\arcsin(x) \leq 2x$, for $0 \leq x \leq \pi/3$. Since $\varphi_k > 1/(m-1)$, for $k \geq 1$, f_{φ_k} is monotonely increasing in ρ, for all $1 \leq k \leq \lceil m/2\rceil - 1$, but not necessarily for $k = 0$. We now show that this implies that $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{\lceil m/2\rceil-1}$. Let $0 \leq k \leq \lceil m/2\rceil - 2$. Since $\pi > \varphi_{k+1} > \varphi_k > 0$, we have, for $0 \leq k \leq \lceil m/2\rceil - 2$, $-\cos\varphi_{k+1} > -\cos\varphi_k$ and, hence

$$1/c = f_{\varphi_{k+1}}(\rho_{k+1}) = f_{\varphi_k}(\rho_k) < f_{\varphi_{k+1}}(\rho_k)$$

and as $f_{\varphi_{k+1}}$ is monotonely increasing in ρ, we obtain $\rho_{k+1} < \rho_k$. □

In the following we investigate the ratio ρ_0/ρ_k.

Lemma 4.7 *$\rho_0/\rho_k \geq 1 + 1/(4m^3)$, for all $1 \leq k \leq \lceil m/2 \rceil - 1$.*

Proof: Since by Lemma 4.6 $\rho_1 \geq \rho_k$, for all for all $2 \leq k \leq \lceil m/2 \rceil$, it suffices to show that $\rho_0/\rho_1 \geq 1 + 1/(4m^3)$. Let f be the function

$$f(\varphi, \rho) = \left|\lambda^{m-1}(1-\lambda)\right|^2 = \rho^{2(m-1)}(\rho^2 - 2\rho\cos\varphi + 1).$$

Note that $f(\varphi_0, \rho_0) = f(\varphi_1, \rho_1) = 1/c^2$ and, therefore, $f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) = f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1)$. Now $f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) = 2\rho_0^{2m-1}(\cos\varphi_0 - \cos\varphi_1)$ and

$$f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1) = \int_{\rho_1}^{\rho_0} \frac{\partial}{\partial\rho} f(\varphi_1, \rho) d\rho \leq (\rho_0 - \rho_1) \max_{\rho \in [\rho_1, \rho_0]} \frac{\partial}{\partial\rho} f(\varphi_1, \rho).$$

If we consider the derivative of f with respect to ρ, then

$$\frac{\partial}{\partial\rho} f(\varphi_1, \rho) = 2m\rho^{2m-3}\left(\rho^2 - 2\frac{2m-1}{2m}\rho\cos\varphi_1 + \frac{2(m-1)}{2m}\right).$$

Hence,

$$\begin{aligned} f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) &= 2\rho_0^{2m-1}(\cos\varphi_0 - \cos\varphi_1) = f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1) \\ &\leq (\rho_0 - \rho_1) \max_{\rho \in [\rho_1, \rho_0]} 2m\rho^{2m-3}\left(\rho^2 - \frac{2m-1}{m}\rho\cos\varphi_1 + \frac{2(m-1)}{2m}\right) \\ &\leq (\rho_0 - \rho_1) 2m\rho_0^{2m-3}(\rho_0 + 1)^2 \end{aligned}$$

and, thus,

$$\frac{\rho_0}{\rho_1} \frac{\rho_0(\cos\varphi_0 - \cos\varphi_1)}{m(\rho_0+1)^2} \leq \frac{\rho_0}{\rho_1} - 1$$

or

$$\frac{\rho_0}{\rho_1} \geq \frac{1}{1 - \rho_0(\cos\varphi_0 - \cos\varphi_1)/(m(\rho_0+1)^2)} \geq 1 + \frac{\rho_0(\cos\varphi_0 - \cos\varphi_1)}{m(\rho_0+1)^2}.$$

In order to bound $\rho_0(\cos\varphi_0 - \cos\varphi_1)/(m(\rho_0+1)^2)$ from below, we need upper and lower bounds for ρ_0. We first give an upper bound. Observe that

$$\begin{aligned} \frac{1}{c} &= \lambda^{m-1}(1-\lambda) = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^{m-1} \frac{\sin\varphi_0}{\sin m\varphi_0} = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^{m} \frac{\sin\varphi_0}{\sin(m-1)\varphi_0} \\ \Rightarrow \quad \rho_0^m &= \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^m = \frac{\sin(m-1)\varphi_0}{\sin\varphi_0 c} \leq \frac{m-1}{c}. \end{aligned}$$

Hence, $\rho_0 \leq \sqrt[m]{(m-1)/c} \leq 1$ since $c \geq 3$.

Now we bound ρ_0 from below. Note that $|1 - \lambda_0|$ is the distance between the point $(1,0)$ and the point λ_0 in the complex plane. Since λ_0 belongs to the wedge S_0 of numbers whose polar angle is in $[0, \pi/3]$ and whose radius is less than one, it is easy to see that the origin is the furthest point in S_0 from $(1,0)$ and $|1 - \lambda_0| \leq 1$. Hence,

$\rho_0 \geq \sqrt[m-1]{1/(|1-\lambda_0|c)} \geq \sqrt[m-1]{1/c}$. Since we assume that $c < m^m/(m-1)^{m-1} < em$, we obtain, $\rho_0 \geq \sqrt[m-1]{1/(em)} \geq 1/3$.

Next we give a lower bound for $\cos\varphi_0 - \cos\varphi_1$. Since $\varphi_0 \in [0, \pi/m]$ and $\varphi_1 \in [2\pi/(m-1), 3\pi/m]$ both of which are contained in $[0, \pi]$, for $m \geq 3$, $\cos\varphi_0 - \cos\varphi_1 \geq \cos\pi/m - \cos 2\pi/(m-1)$. Moreover, since cosine is concave over $[0, \pi/2]$ and $2\pi/(m-1) \leq \pi/2$, for $m \geq 5$,

$$\cos\varphi_0 - \cos\varphi_1 \geq \cos\frac{\pi}{m} - \cos\frac{2\pi}{m-1} \geq \sin\frac{\pi}{m}\left(\frac{2\pi}{m-1} - \frac{\pi}{m}\right) \geq \frac{\pi}{2m}\frac{\pi}{m} \geq \frac{\pi^2}{2m^2},$$

for $m \geq 5$. On the other hand, if $m = 3$, then $\cos(\pi/3) - \cos(2\pi/2) > 1 > \pi^2/18$ and if $m = 4$, then $\cos(\pi/4) - \cos(2\pi/3) > 1/\sqrt{2} > \pi^2/32$, so that the inequality $\cos\varphi_0 - \cos\varphi_1 \geq \pi^2/(2m^2)$ holds for all $m \geq 3$.

Hence, for $1 \leq k \leq \lceil m/2 \rceil$,

$$\frac{\rho_0}{\rho_k} \geq \frac{\rho_0}{\rho_1} \geq 1 + \frac{\pi^2}{6m^3(1+1)^2} \geq 1 + \frac{1}{4m^3}.$$

□

4.3 The Coefficients

We finally give an upper bound on the radius of the coefficients. Recall that the solution of Recurrence Equation 8 is given by

$$y_k = a_0\lambda_0^k + a_1\lambda_1^k + \cdots + a_{m-1}\lambda_{m-1}^k.$$

Let $A = (\lambda_j^i)_{0\leq i,j\leq m-1}$, $\overline{a} = (a_0, \ldots, a_{m-1})$, and $\overline{D} = (D, \ldots, D)$. The coefficients a_i are the solution of the linear equation system $A\overline{a} = \overline{D}$. Let $A_i(x)$ the matrix A where the ith column is replaced by the vector $(x, \ldots, x)^T$. By Cramer's rule a_i is given as

$$a_i = \frac{\det(A_i(D))}{\det(A)} = D\frac{\det(A_i(1))}{\det(A)} = D\frac{\prod_{j=0,j\neq i}^{m-1}(1-\lambda_j)}{\prod_{j=0,j\neq i}^{m-1}(\lambda_i - \lambda_j)} \tag{16}$$

since both A and $A_i(1)$ are Vandermonde matrices.

In order to bound the size of the ratio of $|a_i/a_0|$ we have the following lemma.

Lemma 4.8 $|a_i/a_0| \leq 4^{2m}m^m$.

Proof: We have

$$\begin{aligned}\left|\frac{a_i}{a_0}\right| &= \left|\frac{1-\lambda_i}{1-\lambda_0}\right|\left|\frac{\prod_{j=0,j\neq 0}^{m-1}(\lambda_0-\lambda_j)}{\prod_{j=0,j\neq i}^{m-1}(\lambda_i-\lambda_j)}\right| \leq \frac{1+|\lambda_i|}{|1-\lambda_0|}\frac{\prod_{j=0,j\neq 0}^{m-1}(|\lambda_0|+|\lambda_j|)}{\prod_{j=0,j\neq i}^{m-1}|\lambda_i-\lambda_j|} \\ &\leq \frac{2}{|1-\lambda_0|}\frac{2^{m-1}}{\prod_{j=0,j\neq i}^{m-1}|\lambda_i-\lambda_j|}.\end{aligned}$$

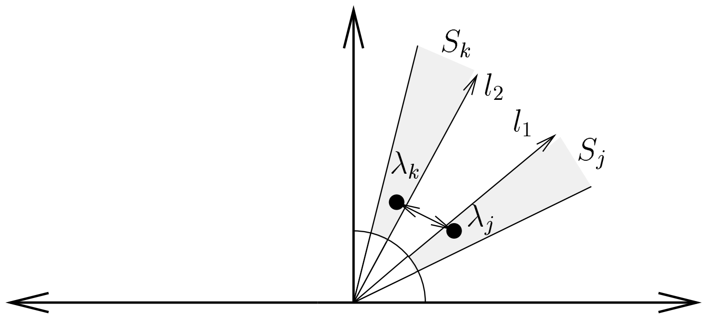

Figure 2: The sectors that λ_k and λ_j belong to.

In order to obtain an upper bound for $1/|1-\lambda_0|$ we observe that

$$1/|1-\lambda_0| \quad = \quad c\,|\lambda_0^{m-1}| \;\leq\; c \;\leq\; em \tag{17}$$

Finally, we give a lower bound for $|\lambda_k - \lambda_j|$. We first observe that since $|\lambda_i|^{m-1} \geq 1/(c|1-\lambda_i|) \geq 1/(2c) \geq 1/(2em)$, $|\lambda_i| \geq \sqrt[m-1]{1/(2em)} \geq 1/5$.

If we view λ_k and λ_j as two points in the complex plane, then λ_k is contained in the angular sector of $S_k = [2k\pi/(m-1), (2k+1)\pi/m]$ and λ_j is contained in the angular sector of $S_j = [2j\pi/(m-1), (2j+1)\pi/m]$ (see Figure 2). Since $|\lambda_k| \geq 1/5$ and $|\lambda_j| \geq 1/5$, the distance between λ_k and λ_j is at least the distance between the points of S_k and S_j outside the circle through the origin with radius $1/5$. W.l.o.g. assume that $k > j$. Let l_1 be the line with angle $2k\pi/(m-1)$ through the origin and l_2 be the line with angle $(2j+1)\pi/m$ through the origin. If p is the point on l_1 with distance $1/5$ to the origin, then the distance of S_k to S_j outside the circle with radius $1/5$ is at most the distance of p to l_2. By elementary geometry we obtain that

$$|\lambda_k - \lambda_j| \geq d(p, l_2) \quad = \quad \frac{\sin\left(2k\pi/(m-1) - (2j+1)\pi/m\right)}{5} \geq \frac{\pi}{10m} \geq \frac{1}{4m}. \tag{18}$$

Combining the estimates for $|1-\lambda_0|$ and $|\lambda_k - \lambda_j|$ we obtain

$$\left|\frac{a_i}{a_0}\right| \quad \leq \quad \frac{2^m}{|1-\lambda_0|\prod_{j=0,j\neq i}^{m-1}|\lambda_i - \lambda_j|} \leq 2^m em(4m)^{m-1} \leq 4^{2m}m^m$$

as claimed. $\square$

The following lemma gives a lower bound of the absolute value of a_0.

Lemma 4.9 $|a_0| > D/(2em)^{m-1}$.

Proof: The proof follows easily from Equations 16 and 17.

$$|a_0| = D\frac{\prod_{j=1}^{m-1}|1-\lambda_j|}{\prod_{j=1}^{m-1}|\lambda_0 - \lambda_j|} \geq D\frac{(1/em)^{m-1}}{2^{m-1}}.$$

Note that the lower bound for $|1-\lambda_0|$ of Equation 17 is also a lower bound for $|1-\lambda_i|$ and that $|\lambda_0 - \lambda_j| \leq \rho_0 + \rho_j < 2$. $\square$

4.4 Putting it all Together

We now put the estimates we obtained for the radii and the angles of the roots of Equation 8 as well as the coefficients into use. W.l.o.g. we assume that m is even. If m is odd, then an analogous proof works. We start off by proving a lower and an upper bound on the size of y_k.

Lemma 4.10

$$\cos(\theta_0 + k\varphi_0) - \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k} \;\leq\; \frac{y_k}{2|a_0|\rho_0^k} \;\leq\; \cos(\theta_0 + k\varphi_0) + \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k}.$$

Proof: Recall that

$$y_k \;=\; \sum_{j=0}^{\lfloor m/2 \rfloor} a_j \lambda_j^k + \overline{a}_j \overline{\lambda}_j^k \leq a_0 \lambda_0^k + \overline{a}_0 \overline{\lambda}_0^k + \sum_{j=0}^{\lfloor m/2 \rfloor} 2Re(a_j \lambda_j^k).$$

If $\lambda_0 = \rho_0 e^{i\varphi_0}$ and $a_0 = \sigma_0 e^{i\theta_0}$, then

$$a_0 \lambda_0^k + \overline{a}_0 \overline{\lambda}_0^k \;=\; \sigma_0 \rho_0^k e^{i(\theta_0 + k\varphi_0)} + \sigma_0 \rho_0^k e^{-i(\theta_0 + k\varphi_0)} = 2\sigma_0 \rho_0^k \cos(\theta_0 + k\varphi_0).$$

and

$$\frac{y_k}{2|a_0|\rho_0^k} \;\leq\; \cos(\theta_0 + k\varphi_0) + \sum_{j=0}^{\lfloor m/2 \rfloor} \left| \frac{a_j}{a_0} \right| \frac{\rho_j^k}{\rho_0^k} \;\leq\; \cos(\theta_0 + k\varphi_0) + \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k}$$

by Lemmas 4.7 and 4.8. Similarly,

$$\frac{y_k}{2|a_0|\rho_0^k} \;\geq\; \cos(\theta_0 + k\varphi_0) - \sum_{j=0}^{\lfloor m/2 \rfloor} \left| \frac{a_j}{a_0} \right| \frac{\rho_j^k}{\rho_0^k} \;\geq\; \cos(\theta_0 + k\varphi_0) - \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k}.$$

$\square$

In the following we show that if

$$c < \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then there is a step k_0 such that $y_{k_0 - 1} > c^2$ and $y_{k_0 + 2} < 0$, which proves Lemma 2.5, for $m \geq 3$.

In the following let $\varepsilon = \sqrt{m^m/(m-1)^{m-1} - c}$. We assume that $\varepsilon < 1$. The case $\varepsilon \geq 1$ can be treated as the case $c \leq 3$ in the case $m = 2$. Let n_0 be the first index greater than $4m^3(3m \log m - \log \varepsilon) + 1$ such that

$$\cos(\theta_0 + n_0 \varphi_0) > 0 \qquad \text{and} \qquad \cos(\theta_0 + (n_0 + 1)\varphi_0) \leq 0.$$

Since the distance between two consecutive transitions from positive to negative values of cosine is at most 2π and $n_0 \geq 4m^3(3m\log m - \log\varepsilon) + 1$, we have that $n_0 - 4m^3(3m\log m - \log\varepsilon) - 1 \leq 2\pi/\varphi_0$ and

$$n_0 \leq 4m^3(3m\log m - \log\varepsilon) + 1 + \frac{2\pi}{\varphi_0} \leq 4m^3(3m\log m - \log\varepsilon) + 1 + \frac{2\pi m^{3/2}}{\varepsilon}. \quad (19)$$

Note that since $\varepsilon \leq 1$, $\varepsilon/m^{3/2} < 1/\sqrt{3}m$ and $\varphi_0 \geq \varepsilon/m^{3/2}$ by Lemma 4.5. Once we have chosen n_0, the values of y_{n_0-1} and y_{n_0+2} are bounded as follows.

Lemma 4.11

$$y_{n_0-1} \geq 2|a_0|\rho_0^{n_0-1}\frac{\varphi_0}{4} \qquad \textit{and} \qquad y_{n_0+2} \leq -2|a_0|\rho_0^{n_0+2}\frac{\varphi_0}{4}.$$

Proof: We first observe that if $n_0 > 4m^3(3m\log m - \log\varepsilon) + 1$, then

$$n_0 - 1 \geq \frac{3m\log m - \log\varepsilon}{\log(1+1/(4m^3))} \geq \frac{(m+1)\log m + \log(4m+2) + \log(m^{3/2}/\varepsilon)}{\log(1+1/(4m^3))} \quad (20)$$

where we use $\log(1+x) \leq x$. Inequality 20 now implies that

$$\left(1+\frac{1}{4m^3}\right)^{n_0-1} \geq \frac{4^{2m+1}m^{m+1}}{\varphi_0} \qquad \text{and} \qquad \frac{4^{2m}m^{m+1}}{(1+1/4m^3)^{n_0-1}} \leq \frac{\varphi_0}{4}.$$

By Lemma 4.10

$$\begin{aligned}\frac{y_{n_0-1}}{2|a_0|\rho_0^{n_0-1}} &\geq \cos(\theta_0 + (n_0-1)\varphi_0) - \frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^{n_0-1}} \\ &\geq \cos(\pi/2 - \varphi_0) - \frac{\varphi_0}{4} \geq \frac{\varphi_0}{4}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{y_{n_0+2}}{2|a_0|\rho_0^{n_0+2}} &\leq \cos(\theta_0 + (n_0+2)\varphi_0) + \frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^{n_0}} \\ &\leq \cos(\pi/2 + \varphi_0) + \frac{\varphi_0}{4} \leq -\frac{\varphi_0}{4}\end{aligned}$$

as claimed. □

With the above preparations we now can prove the main lemma.

Lemma 4.12 *If*

$$c < \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then $y_{n_0-1} > c^2$ *and* $y_{n_0+2} < 0$.

Proof: Inequality $y_{n_0+2} < 0$ follows directly from Lemma 4.11. Hence, we only have to show that $y_{n_0-1} > c^2$.

Step 1 We first show that a lower bound on D. If

$$c \;<\; \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then

$$\begin{aligned}
\log D &> \frac{22m^4 \log m}{\sqrt{m^m/(m-1)^{m-1} - c}} = \frac{22m^4 \log m}{\varepsilon} \\
&\geq (12m^4 \log m - 4m^3 \log \varepsilon) \log 3 + \frac{m^4 \log m}{\varepsilon} \\
&\geq \left(4m^3(3m \log m - \log \varepsilon) + \frac{2\pi m^{3/2}}{\varepsilon}\right) \log 3 + \log(2em)^{m-1} + \log((em)^2 m^{3/2})
\end{aligned}$$

and, therefore,

$$D \;>\; \frac{3^{n_0-1}(em)^2(2em)^{m-1}m^{3/2}}{\varepsilon} > \frac{3^{n_0-1}c^2(2em)^{m-1}m^{3/2}}{\varepsilon} \tag{21}$$

since by Equation 19

$$n_0 - 1 \leq 4m^3(3m \log m - \log \varepsilon) + \frac{2\pi m^{3/2}}{\varepsilon}.$$

Step 2 We now show that $y_{n_0-1} > c^2$. We have by

$$\begin{aligned}
y_{n_0-1} &\overset{\text{(Lemma 4.11)}}{\geq} 2|a_0|\rho_0^{n_0-1}\frac{\varphi_0}{4} \overset{\text{(Lemma 4.9)}}{\geq} 2\frac{D}{(2em)^{m-1}}\rho_0^{n_0-1}\frac{\varphi_0}{4} \\
&\overset{(\rho_0 \geq 1/3)}{\geq} \frac{2D}{(2em)^{m-1}}(1/3)^{n_0-1}\frac{\varphi_0}{4} \overset{\text{(Lemma 4.5)}}{\geq} \frac{D\varepsilon}{3^{n_0-1}(2em)^{m-1}m^{3/2}} \overset{\text{(Equation 21)}}{>} c^2
\end{aligned}$$

as claimed. □

Now that we have shown Lemma 2.5 we can reformulate Theorem 2.6 in the following way.

Theorem 4.13 *There is no search strategy for a target on m rays which is contained in the interval $[1, D(m-1)2e/(m-2)]$ with a competitive ratio of less than*

$$1 + 2\left(\frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D}\right).$$

5 An Asymptotically Optimal Strategy

After having proven a lower bound for searching on m rays with an upper bound on the target distance, the questions remains whether this is the best bound possible. In this section we present a strategy to search on m rays that achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ even if the maximum distance D of the target to the starting point is unknown. Hence, the lower bound proven in the previous section cannot be improved if we consider the convergence rate as D increases to infinity.

The strategy $X = (x_1, x_2, \dots)$[1] that achieves a competitive ratio of $1+2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ is given by

$$x_i = \sqrt{1+\frac{i}{m}}\left(\frac{m}{m-1}\right)^i.$$

The competitive ratio of Strategy X in Step $k+m$ is bounded by $1+2c$ where

$$\begin{aligned} c &\geq \frac{\sum_{j=1}^{k+m-1}\sqrt{1+\frac{j}{m}}\left(\frac{m}{m-1}\right)^j}{\sqrt{1+\frac{k}{m}}\left(\frac{m}{m-1}\right)^k} \\ &= \sum_{j=1}^{k-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} + \sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^j, \end{aligned}$$

for $k \geq 1$. We present an upper bound for the sums on the right hand side. We first consider the sum

$$\sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^j.$$

The Taylor-expansion of $\sqrt{1+x}$ yields $\sqrt{1+x} \leq 1 + x/2$, for $x \leq 1$, and, therefore,

$$\begin{aligned} \sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^j &\leq \sum_{j=0}^{m-1}\left(1+\frac{1}{2}\frac{j}{k+m}\right)\left(\frac{m}{m-1}\right)^j \\ &= \frac{m^m}{(m-1)^{m-1}} - (m-1) + \frac{(m-1)m}{k+m}. \qquad (22) \end{aligned}$$

Now we consider the sum

$$\sum_{j=1}^{k-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} = \sum_{j=1}^{k-1}\sqrt{1-\frac{j}{k+m}}\left(\frac{m-1}{m}\right)^j. \qquad (23)$$

Similar to above we observe that

$$\sqrt{1-x} \leq 1 - \frac{1}{2}x - \frac{1}{8}x^2,$$

[1] For convenience we start with x_1 instead of x_0.

for $x \leq 1$, and, therefore,

$$
\begin{aligned}
& \sum_{j=1}^{k-1} \sqrt{1 - \frac{j}{k+m}} \left(\frac{m-1}{m}\right)^j \\
\leq \;& \sum_{j=1}^{k-1} \left(1 - \frac{1}{2}\frac{j}{k+m} - \frac{1}{8}\left(\frac{j}{k+m}\right)^2\right)\left(\frac{m-1}{m}\right)^j \\
= \;& m - 1 - m\left(\frac{m-1}{m}\right)^k + \frac{m(m-1) - (k-m-1)m\left(\frac{m-1}{m}\right)^k}{k+m} + \\
& \frac{m(m-1)(2m-1)}{(k+m)^2} - \frac{(k^2 + 2k(m-2) + 2m^2 - 3m + 1)m\left(\frac{m-1}{m}\right)^k}{(k+m)^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \sum_{j=0}^{m-1} \sqrt{1 + \frac{j}{k+m}} \left(\frac{m}{m-1}\right)^j + \sum_{j=1}^{k-1} \sqrt{\frac{j+m}{k+m}} \left(\frac{m-1}{m}\right)^{k-j} \\
\leq \;& \frac{m^m}{(m-1)^{m-1}} - \frac{1}{8}\frac{m(m-1)(2m-1)}{(k+m)^2} \\
& + \left(\frac{1}{2}\frac{k-m-1}{k+m} + \frac{1}{8}\frac{k^2 + 2k(m-2) + 2m^2 - 3m + 1}{(k+m)^2} - 1\right) m \left(\frac{m-1}{m}\right)^k \\
\leq \;& \frac{m^m}{(m-1)^{m-1}} - \frac{1}{8}\frac{m(m-1)(2m-1)}{(k+m)^2}
\end{aligned}
$$

since

$$\frac{1}{2}\frac{k-m-1}{k+m} + \frac{1}{8}\frac{k^2 + 2k(m-2) + 2m^2 - 3m + 1}{(k+m)^2} \leq 1.$$

There are two special cases $k = 1$ and $k \leq 0$ that have to be considered separately. If $k = 1$, then Sum (23) is 0 and Sum (22) adds up to

$$\frac{m^m}{(m-1)^{m-1}} - (m-1) + \frac{(m-1)m}{2(1+m)} \quad \leq \quad \frac{m^m}{(m-1)^{m-1}} - \frac{m-1}{2}.$$

If $-m + 1 \leq k \leq 0$, then the target is discovered during the first m iterations. The worst case occurs if the first $m-1$ rays are explored and then the target is detected on the mth ray at a distance of $1 + \varepsilon$, for some $\varepsilon > 0$. The competitive ratio is bounded by

$$1 + 2\sum_{j=1}^{m-1} \sqrt{1 + \frac{j}{m}} \left(\frac{m}{m-1}\right)^j \quad \leq \quad \frac{m^m}{(m-1)^{m-1}} - \frac{m+1}{2}.$$

Finally, we relate the number of steps $k + m$ to the distance D to the target. If the target is detected in Step $k + m$, then the distance D to s is in the interval

$[\sqrt{1+\frac{k}{m}}(m/(m-1))^k, \sqrt{1+\frac{k+m}{m}}(m/(m-1))^{k+m}]$ and D is bounded from below by

$$\sqrt{1+\frac{k}{m}}\left(\frac{m}{m-1}\right)^k \leq D$$

or

$$\frac{1}{2}\log(1+k/m) + k\log\left(1+\frac{1}{m-1}\right) \leq \log D$$

which implies

$$k \leq \frac{\log D}{\log\left(1+\frac{1}{m-1}\right)} \leq (m-1)\log D.$$

Hence,

$$\begin{aligned} 1+2\frac{\sum_{j=1}^{k+m-1}\sqrt{1+\frac{j}{m}}\left(\frac{m}{m-1}\right)^j}{\sqrt{1+\frac{k}{m}}\left(\frac{m}{m-1}\right)^k} &\leq 1+2\left(\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{8(\log D + m/(m-1))^2}\right) \\ &\leq 1+2\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{4\log^2(3D)}. \end{aligned}$$

We have shown the following theorem.

Theorem 5.1 *There is a strategy that achieves a competitive ratio of at most*

$$1+2\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{4\log^2(3D)}$$

if the target is placed at distance $D > 1$ to s.

By Theorem 4.13 the strategy we have presented above is optimal (up to a constant) if D goes to infinity.

6 Computing the Optimal Strategy

In this section we present an algorithm to compute the optimal strategy to search on m bounded rays. As opposed to the previous sections we now assume that we are given the competitive ratio $1+2c$ and we want to compute the maximal *reach* for $1+2c$ [IKL97]. Recall that the *reach* of a strategy X is the maximum distance D such that a target placed at a distance D to the origin is still detected by a robot using X if the competitive ratio of X equals $1+2c$. Note that once we are able to compute the maximum reach, we can easily compute the minimal competitive ratio for a given D by applying binary search. This only increases the running time proportional to the number of bits necessary to represent D.

In the case $m = 2$ it is not too hard to derive a recurrence equation for the optimal reach (see [IKL97]). As in the proof of Lemmas 2.2 and 2.3 we can show that there exists a strategy with maximal reach that is periodic and satisfies the equations[2]

$$\sum_{i=0}^{k+m-1} x_i = c\,x_k, \quad (2) \qquad \text{and} \qquad \sum_{i=0}^{m-2} x_i \;\leq\; c \quad (3).$$

In fact, if $\sum_{i=0}^{m-2} x_i < c$, then there is a $\lambda > 1$ such that the strategy λX satisfies Equation 2 and $\sum_{i=0}^{m-2} \lambda x_i = c$. If D is the reach of X, then $\lambda D > D$ is the reach of λX. Hence, we can assume that we have equality in Equation 3 for a strategy with maximal reach. For $m = 2$, this implies that $x_0 = c$ and x_k is given by

$$x_k = c\,x_{k-1} - \sum_{i=0}^{k-1} x_i,$$

for $k \geq 1$, which determines the strategy completely. For $m > 2$, we still have equality in Equation 3 but now we only obtain the *sum* of the first $m-1$ steps lengths $x_0, \ldots, x_{m-2}$ instead of their values.

Hence, we take a different approach. Let Y again be the sequence defined by $y_i = x_{n-i-1}$, for $0 \leq i \leq n-m$. It satisfies recurrence equation (7), namely

$$y_{k+m} - y_{k+m-1} + \frac{1}{c} y_k \;=\; 0,$$

for all $0 \leq k \leq n-m$. However, we only have the initial values for $y_0, \ldots, y_{m-2}$ which are equal to D. Since we need one more initial value we set $\alpha = y_{m-1}/D$, where $0 < \alpha < 1$. Let $Y(c, D, \alpha D) = (y_0, y_1, \ldots)$ be the infinite sequence that is given by Equation 7 and the above initial values. If $c < c_m$, then there is an index n_0 such that y_{n_0} is negative. By Equation 19

$$n_0 \leq 4m^3(3m\log m - \log\varepsilon) + 1 + \frac{2\pi m^{3/2}}{\varepsilon},$$

where $\varepsilon = \sqrt{m^m/(m-1)^{m-1} - c}$. We choose n to be the index such that y_n is minimal among $y_0, \ldots, y_{n_0-1}$. The value y_n is now the lower bound on the distance to the target. If we set $x'_k = y_{n-k-1}/y_n$, for $0 \leq k \leq n$, then we obtain a strategy with a lower bound of $y_n/y_n = 1$ to the target and reach D/y_n, which is obviously the largest possible reach for a strategy that satisfies Equations 2 and 3 with the above initial values. Unfortunately, we know neither D nor α. However, we can set $D = 1$ since we are going to scale by $1/y_n$ later anyway.

Since we do not know α, we consider the values y_k as numbers over the extended field $\mathbb{R}[\alpha] = \{x + y\alpha \mid x, y \in \mathbb{R}\}$, that is, α is treated as formal parameter. Hence,

[2] To see this we just note that if $\sum_{i=0}^{k+m-1} x_i^* < c x_k^*$, for some k, then we can decrease x_k^* by some amount $\varepsilon > 0$ and increase $x_{n-m+1}^*, \ldots, x_n^*$ by ε/m, thus achieving a greater reach.

$y_k = u_k + \alpha\, v_k$, for some values u_k and v_k. On the other hand, $y_k = y_{k-1} - (1/c)\, y_{k-m}$, for $m \leq k \leq n$. This yields two recurrences for u_k and v_k.

$$u_k = u_{k-1} - (1/c)\, u_{k-m}, \qquad \text{and} \qquad v_k = v_{k-1} - (1/c)\, v_{k-m}.$$

The initial values for the sequence $U = (u_0, u_1, \dots)$ are now given by $u_0 = \dots = u_{m-2} = 1$ and $u_{m-1} = 0$. The initial values for the sequence $V = (v_0, v_1, \dots)$ are given by $v_0 = \dots = v_{m-2} = 0$ and $v_{m-1} = 1$. If we stop after n steps, then Equation 3 should be satisfied, that is, we require

$$\sum_{i=n-m+1}^{n-1} u_i + \alpha_n v_i = c(u_n + \alpha_n v_n) \quad \text{or} \quad \alpha_n = -\frac{\sum\limits_{i=n-m+1}^{n-1} u_i - c\, u_n}{\sum\limits_{i=n-m+1}^{n-1} v_i - c\, v_n}. \tag{24}$$

We obtain the following algorithm.

Algorithm Maximal Reach
Input: The competitive ratio $1 + 2c$ and the number of rays m.
Output: An integer n and a strategy $X = (x_0, \dots, x_n)$ such that the reach of X is maximal.

if $c \geq m^m/(m-1)^{m-1}$ **then return** ∞, $x_k = (1 + 1/(m-1))^k$
for $i \leftarrow 0$ **to** $m-2$ **do** let $u_i \leftarrow 1$, $v_i \leftarrow 0$
let $u_{m-1} \leftarrow 0$, $v_{m-1} \leftarrow 1$

let $y_{min} \leftarrow 1$, $n_{min} \leftarrow m$, $\varepsilon \leftarrow (m^m/(m-1)^{m-1} - c)^{1/2}$
let $n_\varepsilon \leftarrow 4m^3(3m \log m - \log \varepsilon) + 1 + 2\pi m^{3/2}/\varepsilon$
for $n \leftarrow m$ **to** n_ε **do**
  let $u_n \leftarrow u_{n-1} - (1/c)\, u_{n-m}$, $v_n \leftarrow v_{n-1} - (1/c)\, v_{n-m}$
  let $\alpha_n \leftarrow -(\sum_{i=n-m+1}^{n-1} u_i - c\, u_n)/(\sum_{i=n-m+1}^{n-1} u_i - c\, v_n)$
  if $(u_n + \alpha_n v_n < y_{min})$
   then let *positive* $\leftarrow$ true
    for $j \leftarrow m$ **to** n **do**
     /* Test if all elements are positive */
     if $u_j + \alpha_n v_j \leq 0$ **then** let *positive* $\leftarrow$ false
    if *positive* **then** $y_{min} \leftarrow y_n$, $n_{min} \leftarrow n$
  end if

let $n \leftarrow n_{min}$
for $i \leftarrow 0$ **to** $n-1$ **do** let $x_i \leftarrow (u_{n-i-1} + \alpha_n v_{n-i-1})/y_{min}$
return n, $(x_0, \dots, x_{n-1}, 1/y_{min})$

We show the correctness of Algorithm *Maximal Reach* in the following two lemmas.

Lemma 6.1 *The competitive ratio of X is $1 + 2c$.*

Proof: First note that because of the test in Step 11 all the elements of X are positive. By the choice of α, U, and V, X obviously satisfies

$$\sum_{i=0}^{m-2} x_i = c \qquad \text{and} \qquad x_{k+m-1} = c(x_k - x_{k-1})$$

for $0 \leq k \leq n-m$ if we set $x_{-1} = 1$.[3] Using induction we see that

$$\sum_{i=0}^{k+m-1} x_i \;=\; x_{k+m-1} + cx_{k-1} \;=\; c(x_k - x_{k-1}) + cx_{k-1} \;=\; cx_k$$

for $0 \leq k \leq n-m$ as claimed. □

Lemma 6.2 *The reach of Strategy X is at least as large as the reach of any other strategy with competitive ratio $1 + 2c$.*

Proof: Let $X^* = (x_0^*, x_1^*, x_2^*, \ldots, x_l)$ be a strategy with maximal reach for competitive ratio $1 + 2c$. By Equation 19 $l \leq n_\varepsilon$. As we observed previously, X^* satisfies the conditions $\sum_{i=0}^{k+m-1} x_i^* = cx_k^*$, for $0 \leq k \leq l-m$, and $\sum_{i=0}^{m-2} x_i^* = c$. We define a sequence $Y^* = (y_0^*, \ldots, y_l^*)$ by $y_i^* = x_{l-i-1}^*/x_l^*$, for $0 \leq i \leq l$, where we set $x_{-1}^* = 1$. The reach of X^* is $x_{l-1}^* = x_{l-1}^*/x_{-1}^* = (x_{l-1}^*/x_l^*)/(x_{-1}^*/x_l^*) = 1/y_l^*$.

The sequence Y^* satisfies recurrence equation 7. By a simple induction it can be easily seen that $y_k^* = u_k + y_{m-1}^* v_k$. Because of Equations 3 and 24 we obtain that $y_{m-1}^* = \alpha_l$. Hence, Y^* is computed in Step 10 if $k = l$. Let Y be the sequence computed by Algorithm *Maximal Reach*. Since y_n is chosen to be minimal, $y_n \leq y_l^*$. Hence, the reach $1/y_n$ of X is at least as large as the reach of X^*. □

If c consists of b bits, then the time complexity of Algorithm *Maximal Reach* is quadratic in $n_\varepsilon = \sqrt{2^b}$, that is, $\Theta(2^b)$. On the other hand, since $\log D = \Omega(n_\varepsilon)$,[4] the time complexity is quadratic in the size of the output.

We have implemented Algorithm *Maximal Reach* in Maple. In Figure 3a the maximal reach of the optimal strategies for different values of m is shown. The figure illustrates nicely that the logarithm of the maximal reach depends linearly on $1/\varepsilon$. In Figure 3b we compare the maximal reach of the optimal strategy to the maximal reach of the strategy presented in Section 5 and the strategy given by $x_k = (3/2)^k$ for $m = 3$. It can be seen that the maximal reach of the optimal strategy increases much faster than that of the other two strategies. The figure also shows that the maximal reach of the strategy presented in Section 5 is a linear function of $1/\varepsilon$ whereas the maximal reach of the other strategy is a logarithmic function of $1/\varepsilon^2$. It should be noted that the lower bound we have presented—which is now an upper bound on the maximal reach—does not fit into the figure as it starts at a value of > 2000 and has a much steeper slope.

[3]Note that y_n is chosen to satisfy $y_n = y_{n-1} - 1/cy_{n-m}$. Since we divide by y_n, this implies that $y_{n-m}/y_n = c(y_{n-1}/y_n - 1)$ or $x_{m-1} = c(x_0 - x_{-1})$.

[4]This follows from the fact that there is a strategy with such that $\log D$ is in $\Omega(1/\varepsilon)$—as, for instance, the strategy presented in Section 5.

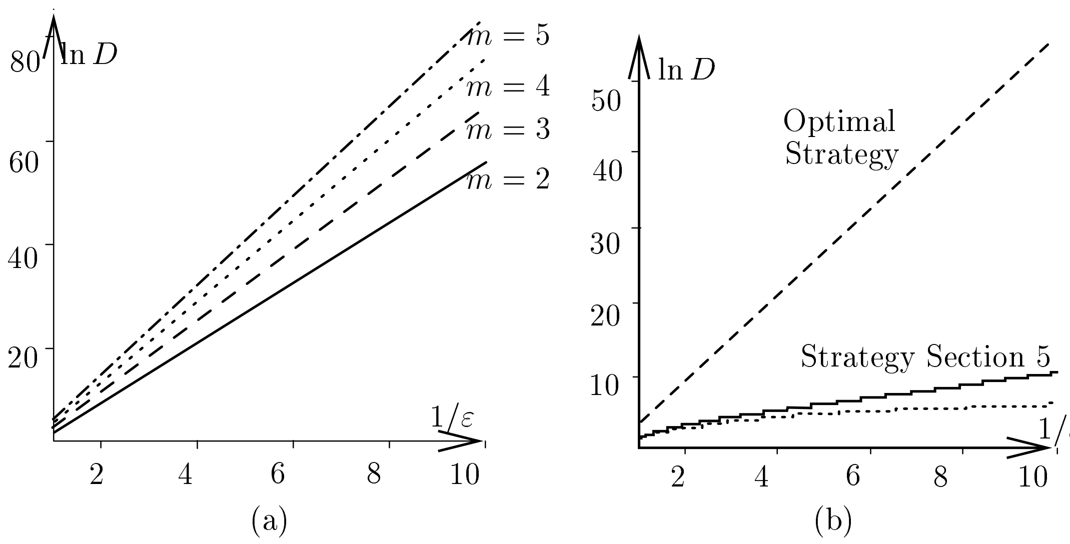

Figure 3: (a) The reach of the optimal strategy for different values of m. (b) The reach of the optimal strategy for $m = 3$ compared to the reach of the strategy proposed in Section 5 and the strategy given by $x_k = (3/2)^k$.

7 Conclusions

We present a lower bound for the problem of searching on m concurrent rays if an upper bound D on the maximal distance to the target is given. We show that in this case the competitive ratio of a search strategy is at least $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$. Our approach is based on deriving a recursive equation for the step length in each iteration of an optimal strategy. The recursive equation gives rise to a characteristic equation whose roots determine the properties of the strategy. By computing upper and lower bounds on the radii and polar angles of the roots in polar coordinates we can show that the competitive ratio has to be sufficiently large if the target is far away.

We also present a strategy which achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ if the target is detected at distance D. The strategy does not need to know an upper bound on D in advance and still achieves the same convergence rate as the lower bound that we have shown. This implies that the convergence rate of our lower bound is tight (up to a constant that depends on m).

Finally, we present an algorithm to compute the strategy with maximal reach for a given competitive ratio and general m. Our algorithm needs time proportional to the size of the output and exponential in the size of the input.

An interesting open problem is to prove similar results for randomized strategies. One of the problems with randomized strategies is that there is no published proof that there is an optimal periodic strategy. This seems to be a necessary step before the bounded distance problem can be attacked.

References

[BYCR93] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Information and Computation*, 106:234–252, 1993.

[DHS95] A. Datta, Ch. Hipke, and S. Schuierer. Competitive searching in polygons—beyond generalized streets. In *Proc. Sixth Annual International Symposium on Algorithms and Computation*, pages 32–41. LNCS 1004, 1995.

[DI94] A. Datta and Ch. Icking. Competitive searching in a generalized street. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, pages 175–182, 1994.

[Gal80] S. Gal. *Search Games.* Academic Press, 1980.

[Hip94] Ch. Hipke. Online-Algorithmen zur kompetitiven Suche in einfachen Polygonen. Master's thesis, Universität Freiburg, 1994.

[IK95] Ch. Icking and R. Klein. Searching for the kernel of a polygon: A competitive strategy. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 258–266, 1995.

[IKL97] Ch. A. Hipke, Ch. Icking, R. Klein, and E. Langetepe. How to find a point on a line within a fixed distance. Informatik-Bericht 220, Fernuni Hagen, November 1997. Discrete Applied Mathematics, to appear.

[Kle92] R. Klein. Walking an unknown street with bounded detour. *Comput. Geom. Theory Appl.*, 1:325–351, 1992.

[Kle97] R. Klein. *Algorithmische Geometrie.* Addison-Wesley, 1997.

[KMSY94] M.-Y. Kao, Y. Ma, M. Sipser, and Y. Yin. Optimal constructions of hybrid algorithms. In *Proc. 5th ACM-SIAM Sympos. Discrete Algorithms*, pages 372–381, 1994.

[KPY96] E. Koutsoupias, Ch. Papadimitriou, and M. Yannakakis. Searching a fixed graph. In *Proc. 23rd Intern. Colloq. on Automata, Languages and Programming*, pages 280–289. LNCS 1099, 1996.

[KRT97] M. Y. Kao, J. H. Reif, and S. R. Tate. Searching in an unknown environment: An optimal randomized algorithm for the cow-path problem. *Information and Computation*, 131(1):63–80, 1997.

[LO96] A. López-Ortiz. *On-line Searching on Bounded and Unbounded Domains.* PhD thesis, Department of Computer Science, University of Waterloo, 1996.

[LOS95] A. López-Ortiz and S. Schuierer. Going home through an unknown street. In S. G. Akl, F. Dehne, and J.-R. Sack, editors, *Proc. 4th Workshop on Algorithms and Data Structures*, pages 135–146. LNCS 955, 1995.

[LOS96] A. López-Ortiz and S. Schuierer. Generalized streets revisited. In M. Serna J. Diaz, editor, *Proc. 4th European Symposium on Algorithms*, pages 546–558. LNCS 1136, 1996.

[LOS97] A. López-Ortiz and S. Schuierer. Position-independent near optimal searching and on-line recognition in star polygons. In *Proc. 5th Workshop on Algorithms and Data Structures*, pages 284–296. LNCS 1272, 1997.

[PY89] C. H. Papadimitriou and M. Yannakakis. Shortest paths without a map. In *Proc. 16th Internat. Colloq. Automata Lang. Program.*, volume 372 of *Lecture Notes in Computer Science*, pages 610–620. Springer-Verlag, 1989.

[ST85] D. D. Sleator and R. E. Tarjan. Amortized efficiency of list update and paging rules. *Communications of the ACM*, 28:202–208, 1985.

[Yin94] Y. Yin. *Teaching, Learning, and Exploration.* PhD thesis, Department of Mathematics and Laboratory for Computer Science, MIT, 1994.