# Parallel Searching on a Lattice

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**Abstract**

We consider the problem of $k$ robots searching on an integer lattice on the plane. We give a strategy for finding a target at an unknown distance away using $k = 2^j$ searchers, where $j \geq 2$, at a competitive ratio of $n/2^{j-1} + 1$. We give a lower bound for general $k$ of $2n/k$. We also give matching upper and lower bounds for the special case $k = 2$.

## 1 Introduction

Searching for an object on the plane with limited visibility is often modelled by a search on a lattice. In this case it is assumed that the search agent identifies the target upon contact. This is the model traditionally used for search and rescue operations in the high seas where a grid pattern is established and search vessels are dispatched in predetermined patterns to search for the target [9, 17].

An axis parallel lattice induces the Manhattan or $L_1$ metric on the plane. One can measure the distances traversed by the search agent or robot using this metric. Traditionally search strategies are analysed using the competitive ratio used in the analysis of on-line algorithms. For a single robot the competitive ratio is defined as the ratio between the distance traversed by the robot in its search for the target and the length of the shortest path between the starting position of the robot and the target. In other words, the competitive ratio measures the detour of the search strategy as compared to the optimal shortest route.

In 1989, Baeza-Yates et al. [1] proposed a strategy for searching on a lattice with a single searcher with a competitive ratio of $2n + 5 + \Theta(1/n)$ to find a point at an unknown distance $n$ from the origin. This is shown to be optimal. The strategy follows a spiral pattern exploring $d$-balls in increasing order, for all integer $d$. This is illustrated in Figure 1. The axes are marked with dashed lines, with the robot starting from the origin. Not surprisingly this strategy is similar to those used in search and rescue operations [9].

However, in real life a search strategy occurs in the presence of multiple agents, which join the search at different times (and often at different speeds). Coordination of such searches is refered to as a "difficult task" in the search and rescue literature [9, 16].

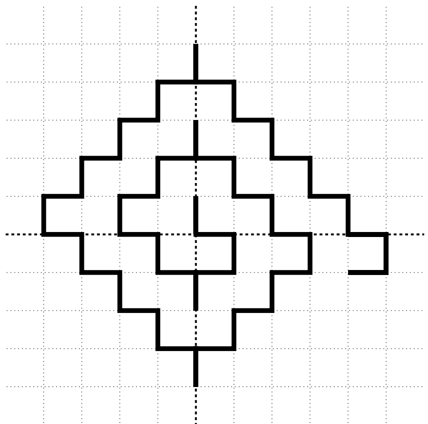

Figure 1: Searching in a lattice.

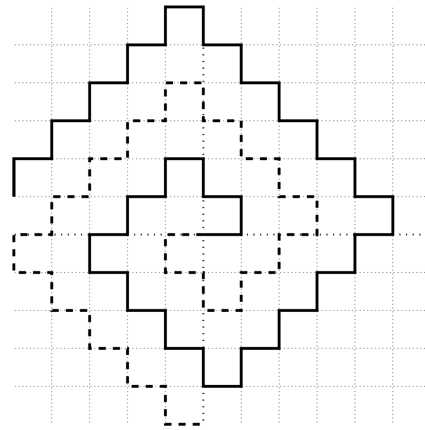

Figure 2: Two robot search.

In this paper we focus on searches with multiple agents or robots starting from the same point (origin). In this context the competitive framework allows then for two natural measures:

1. Time to destination.
2. Total effort.

Time to destination measures the amount of time transcurred between the start of the search and the encounter of the target, in the worst case. In this particular setting most searches benefit from increased number of robots as well as faster robots. Alternatively, total effort compares the total distance traversed by all robots with the shortest path from a start point to the target.

Baeza-Yates and Schott [2] studied parallel searches in the plane where the target is an infinite line at a unit distance from the origin using the total effort metric. In this case they assume that all searchers start from the origin and move at the same speed. Interestingly, under the total effort metric optimality is achieved with four searchers and adding more searchers is of no use.

The time to destination metric is particularly relevant in real-life time-critical situations such as search for survivors. In this situation we have a fixed number of resources that can realistically be brought to bear on the search, and the objetive is to optimally coordinate these many searchers to find the target in the shortest time possible.

In this work we study searches in the lattice using $k$ searchers starting simultaneously from the origin, under the time to destination metric. All $k$ searchers move at the same speed. We give a strategy for finding a target with $k = 2^j$ searchers with a competitive ratio of $n/2^{j-1}+1$ as well as a lower bound for $k$ searchers of $2n/k$ for general $k$.

## 2 Parallel Searching

Consider the case of two robots searching in the lattice. We propose a search path of the lattice as in Figure 2. In this case the robots move in symmetric paths around the origin. In step $i$ the robot explores one quarter of the points at distance $i$ and $i+1$. That is, one side of the $i$ and $i+1$-balls. The $i$-th ball then is explored in four passes. During step $i-1$ each of the two robots explores one quarter of both the $i-1$ and $i$-ball, which covers half of the $i$ ball. In step $i$ each of the two robots explores one quarter of the $i$ and $i+1$ ball. Therefore at the end of this step the $i$-ball is fully explored.

**Theorem 1** *Searching in parallel with two robots for a point at an unknown distance $n$ in the lattice is $n+2$ competitive.*

**Proof.** At step $i$ each robot moves on a zig-zag staircase composed of one side of the points on the $i$ ball together with the immediately neighbouring points on the $i+1$ ball. Note that in the previous step the robot explores one side of the $(i-1)$ and $i$ ball. Therefore at the end of the $i$th step the $i$th ball has been completely explored. The length of this step is $2i+1$. In the worst case, the target point at distance $n$ is the last one explored in the $n$ ball after traversing a path of length $2n$. The total distance traversed is given by $2n+\sum_{i=0}^{n-1}(2i+1) = n^2+2n$ and the competitive ratio is $\mathcal{C} = n+2$ as claimed. $\square$

This is in fact optimal, as the next theorem shows.

**Theorem 2** *Searching in parallel with $k$ robots for a point at an unknown distance $n$ in the lattice requires at least $(2n^2+4n+2-k)/k$ steps.*

**Proof.** Let $A(n)$ be the combined total distance traversed by all robots up and until the last point at distance $n$ is visited. We claim that in the worst case $A(n) \geq 2n^2+4n+2-k$ from which the theorem follows. Define $f(n)$ as the number of points visited on the $(n+1)$-ball before the last visit to a point on the $n$-ball and $g(n)$ as the number of points at a distance greater than $n+1$ before the last point at distance $n$ was explored.

First note that there are $2n^2+2n+1$ points within in the interior of the closed ball of radius $n$ and that visiting any $m$ points requires at least $m-1$ steps. Hence

$$A(n-1) \geq 2(n-1)^2 + 2(n-1) + f(n-1) + g(n-1).$$

Now either $f(n-1) \geq 2n-1$ or $f(n-1) < 2n-1$. If $f(n-1) \geq 2n-1$ we have

$$\begin{aligned} A(n-1) &\geq 2(n-1)^2 + 2(n-1) + (2n-1) + g(n-1) \\ &\geq 2(n-1)^2 + 4(n-1) + 1 \\ &\geq 2(n-1)^2 + 4(n-1) + (2-k) \end{aligned}$$

as claimed. If $f(n-1) < 2n-1$ this means that after the last point at distance $n$ is visited there remain $4n - f(n-1)$ points to visit in the $n$-ball. Now, visiting $m$ points in a ball

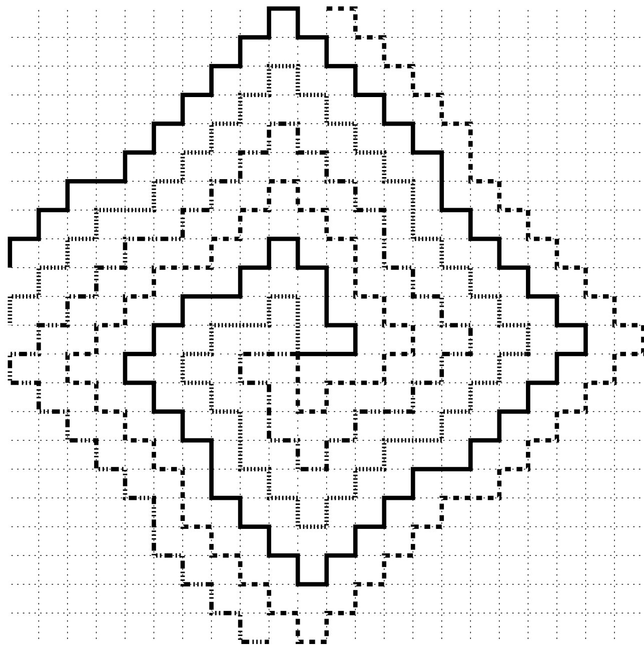

Figure 3: Four robot search.

requires at least $2m-1$ steps with one robot, and $2m-k$ with $k$ robots. Thus visiting the remaining points requires at least $2(4n - f(n-1)) - k$ steps. Hence,

$$\begin{aligned} A(n) &\geq A(n-1)^2 + 2(4n - f(n-1)) - k \\ &\geq 2(n-1)^2 + 2(n-1) + f(n-1) + g(n-1) + 8n - 2f(n-1) - k \\ &\geq 2(n-1)^2 + 2(n-1) - f(n-1) + 8n - k \\ &\geq 2n^2 + 4n + (2-k) \end{aligned}$$

as claimed. In either case, there exist an $n$-ball which is last explored after $2n^2 + 4n + (2-k)$ steps in the worst case. At best each of the robots explores $1/k$th of the total points visited as claimed. $\square$

**Corollary 1** *Searching in parallel for a point at an unknown distance $n$ in the lattice is exactly $n+2$ competitive.*

**Lemma 1** *The lattice can be searched in parallel using four robots at a competitive ratio of $\mathcal{C} = n/2 + 1$.*

**Proof.** The path for four robots is illustrated in Figure 3. Let $L(i)$ be the last point visited at distance $i$. We define step $i$ as the path followed between the $L(i-1)$ and $L(i)$. In step $i$ the robot visits $i + [i \text{ is even}]$ points. Thus each robot visits the last point in the $n$ ball after

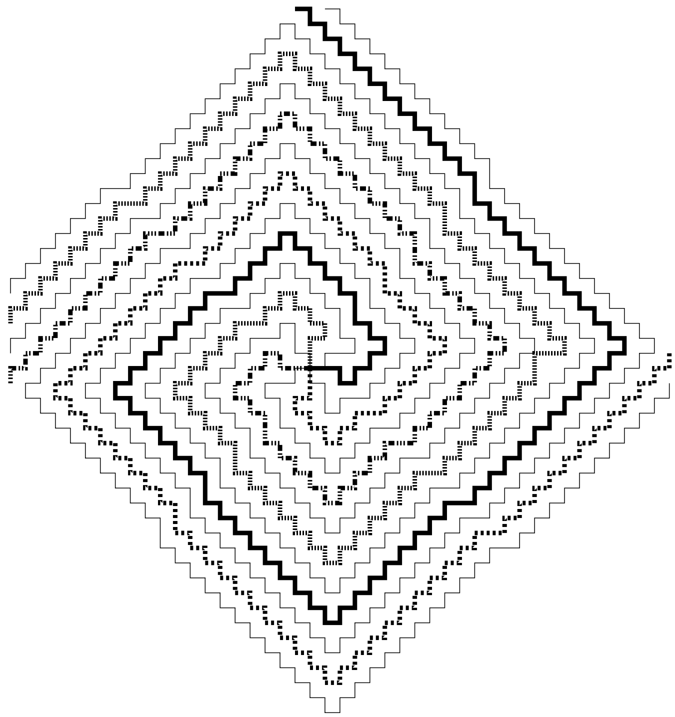

Figure 4: Eight robot search.

traversing a distance of

$$\sum_{i=1}^{n} (i + [i \text{ is even}]) = n(n+1)/2 + \lfloor n/2 \rfloor,$$

for a competitive ratio of $\mathcal{C} = n/2 + 1$. $\square$

Figure 4 shows the path followed by eight robots. In fact, one can infer from the four and eight robot search strategies a pattern to extend a search strategy from $2^j$ robots to $2^{j+1}$ robots. In Figure 5 we overlay the eight robot path (dashed line) on a double-scale four robot path (solid line). The straight edges are replaced by a zig-zag path of the same length in all cases except for the turn points where an extra notch two units larger than the edge in the four robot path is inserted. This path is then rotated 90, 180 and 270 degrees, which defines the path followed by the 2nd, 3rd and 4th robots. The four other robots explore the gap between two consecutive rotated copies of the path.

The competitive ratio can thus be deduced by noticing that in step $i$ a single robot in the eight robot search path traverses twice the distance in the $i$th step of the four robot search strategy, with an extra notch of length two at the turn point. At the same time, the robot reaches twice as far at step $i$ as the four robot path. In total the distance traversed by the robot to reach the last point of the $n$ ball is given by $n^2/4 + n/2$ for even $n$ and $1 + 2\lfloor n/4 \rfloor + 4\left(\lfloor (n+1)/4 \rfloor + \lfloor (n-1)/4 \rfloor^2 + \lfloor (n-1)/4 \rfloor\right)$ for $n$ odd.

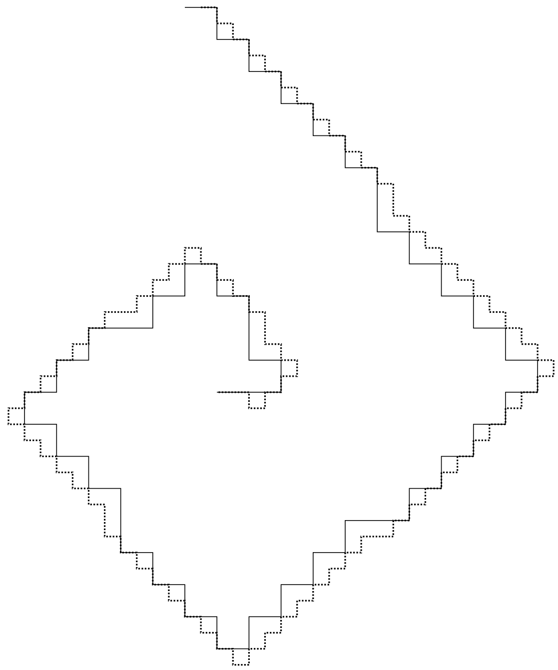

Figure 5: Four robot path (2x scale) and eight robot path (1x scale).

**Lemma 2** *The lattice can be searched in parallel using eight robots at a competitive ratio of* $\mathcal{C} = n/4 + 2$ *which is within an additive factor of* $3/2$ *of the lower bound from Theorem 2.*

**Proof.** Follows from the discussion above. □

**Theorem 3** *The lattice can be searched using* $2^j$ *robots in parallel at a competitive ratio of* $\mathcal{C} = n/2^{j-1} + 1$.

**Proof.** As before, we start from a search path for $2^{j-1}$ robots, which is doubled in scale. Each edge is replaced by a zig-zag of the same length. This creates a gap between two consecutive solutions as the original distance between two path doubled from one unit to two units. In other words the gap is doubled in the scaling. Each of these enlarged gaps is filled by one of the remaining $2^{j-1}$ robots. Let $S(j)$ denote the competitive ratio of searching with $2^j$ robots. Note that at step $k$ the robot explores points at a distance twice as far from the origin as the solution using $2^{j-1}$, hence $S(j) = S(j-1)/2 + 1/4$, where the $1/4$ term is due to the extra notch in each turn point. Thus we obtain $S(j) \leq n/2^{j-1} + 1$ as claimed. □

# 3 Conclusions

We present strategies for searching with two and four robots, which we generalize to strategies for searching with $2^j$ robot with $j \geq 3$. We present a lower bound for $k \geq 2$ robots. The upper and lower bound match to first order terms for $k = 2^j$ robots and it is exact for 2 robots.

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