Constraint Satisfaction Problems: Backtracking Search and Arc Consistency

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Lecture 5

Readings: R & N 6.1, 6.2 & 6.3

Based on work by K. Leyton-Brown, K. Larson, and P. van Beek

Outline

Learning Goals

By the end of the lecture, you should be able to

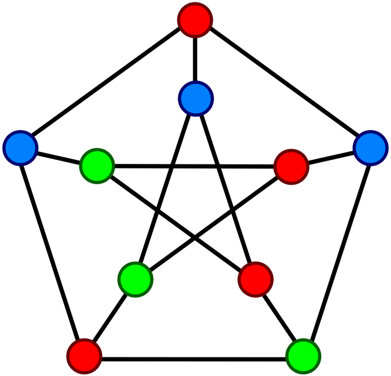
- Formulate a real-world problem as a constraint satisfaction problem.
- Verify whether a variable is arc-consistent with respect to another variable for a constraint.
- Trace the execution of and implement the AC-3 arc consistency algorithm.

- Contrast depth-first search and backtracking search on a CSP.
- Trace the execution of the backtracking search algorithm.
- Trace the execution of the backtracking search algorithm with forward checking and/or arc consistency.
- Trace the execution of the backtracking search algorithm with forward checking and/or arc consistency and with heuristics for choosing variables and values.

Example: Crossword Puzzles

T	A	O	S		S	P	E	C	S		S	E	P	T
E	G	G	Y		A	L	A	A	P		I	L	I	A
T	H	R	E	E	P	E	N	N	Y		D	H	A	K
H	A	E		X	E	B	E	C		S	E	I	N	E
			V	O	L		D	H	O	W	S			
S	W	E	E	N	E	Y		A	N	A	T	M	A	N
P	R	Y	S		S	O	N		E	P	E	I	R	A
L	Y	R	I	C		N	O	S		S	P	A	E	R
I	L	I	C	E	S		R	U	C		P	O	C	K
T	Y	R	A	N	E	D		R	O	P	E	W	A	Y
			T	S	A	R	S		C	U	D			
B	A	G	I	E		O	I	N	K	S		T	S	K
O	L	E	O		S	W	E	E	P	S	T	A	K	E
Y	A	R	N		E	N	V	O	I		A	B	U	T
G	Y	M	S		I	D	E	N	T		G	U	G	A

Example: Graph Coloring Problem



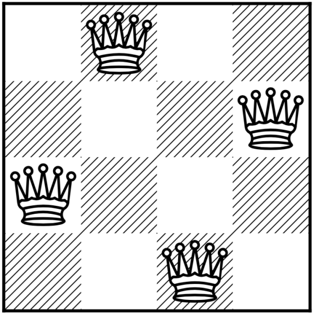
Applications:

- Designing seating plans
- Exam scheduling
- ...

Example: Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Example: 4-Queens Problem



Introduction to CSPs

- So far, search algorithms are unaware of the structure of the states.
- Can we do better by taking advantage of the structure of states?

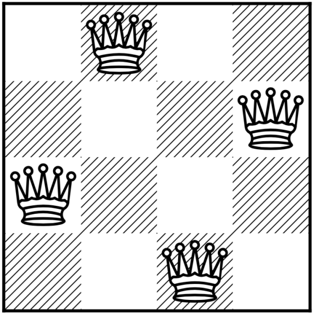
Definition of a CSP

Each state contains

- A set X of variables: $\{X_1, X_2, ..., X_n\}$
- A set D of domains: D_i is the domain for variable X_i, $\forall i$.
- A set C of constraints specifying allowable combinations of values

A solution is an assignment of values to all the variables that satisfy all the constraints.

Example: 4-Queens Problem



CQ: Defining Constraints as a Formula

CQ: How should we encode the following constraint as a propositional formula?

The two queens in columns 0 and 2 are not in the same row or diagonal.

(A) $(x_0 \neq x_2)$

(B) $((x_0 \neq x_2) \land ((x_0 - x_2) \neq 1))$

(C) $((x_0 \neq x_2) \land ((x_0 - x_2) \neq 2))$

(D) $((x_0 \neq x_2) \land (|x_0 - x_2| \neq 1))$

(E) $((x_0 \neq x_2) \land (|x_0 - x_2| \neq 2))$

Solving a CSP - Search and Inference

When solving a CSP, we can combine

- Backtracking search, and
- Inference using the arc-consistency algorithm.

Definition of Arc Consistency

Definition (Arc Consistency)

The variable X_i is arc-consistent with respect to another variable X_j if and only if for every value v_i in D_i, there is a value v_j in D_j such that (v_i, v_j) satisfies the constraint (X_i, X_j).

If X_i is not arc-consistent with the variable X_j, we can make it consistent by removing values in D_i that is not consistent with any value on D_j. This removal can never rule out any solution.

CQ: Definition of Arc Consistency

CQ: Consider the constraint "X is divisible by Y" between two variables X and Y. X is arc-consistent with respect to Y in how many of the four scenarios below?

1. $dom(X) = \{10, 12\}, dom(Y) = \{3, 5\}$
2. $dom(X) = \{10, 12\}, dom(Y) = \{2\}$
3. $dom(X) = \{10, 12\}, dom(Y) = \{3\}$
4. $dom(X) = \{10, 12\}, dom(Y) = \{3, 5, 8\}$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

CQ: Is Arc-Consistency Symmetric?

CQ: True or False:

For every binary constraint C involving variables X and Y,
if X is arc-consistent with respect to Y for C,
then Y is arc-consistent with respect to X for C.

(A) True
(B) False
(C) Not enough information to tell

CQ: Effect of Removing a Value on Arc Consistency

CQ: True or False:

For every binary constraint C involving variables X and Y,
if X is arc-consistent with respect to Y for C,
then, after removing one value from the domain of Y,
X is still arc-consistent with respect to Y for C.

(A) True

(B) False

(C) Not enough information to tell

CQ: Effect of Removing a Value on Arc Consistency

CQ: True or False:

For every binary constraint C involving variables X and Y,
if X is arc-consistent with respect to Y for C,
then, after removing one value from the domain of X,
X is still arc-consistent with respect to Y for C.

(A) True

(B) False

(C) Not enough information to tell

Making (X_i, C) arc-consistent

Let C be a constraint between the variables X_i and X_j.

Algorithm 1 Revise(X_i, C)

revised $\leftarrow$ false
for x in $dom(X_i)$ **do**
    if $\neg\exists\, y \in dom(X_j)$ s.t. (x, y) satisfies the constraint C **then**
        remove x from $dom(X_i)$
        revised $\leftarrow$ true
    end if
end for
return revised

The AC-3 Arc Consistency Algorithm

Algorithm 2 The AC-3 Algorithm

1: Put (v, C) in the set S for every variable v and every constraint involving v.
2: **while** S is not empty **do**
3:     remove (X_i, C_{ij}) from S (C_{ij} is a constraint between X_i and X_j.)
4:     **if** Revise(X_i, C_{ij}) **then**
5:         **if** $dom(X_i)$ is empty **then return** false
6:         **for** X_k where C_{ki} is a constraint between X_k and X_i **do**
7:             add (X_k, C_{ki}) to S
8:         **end for**
9:     **end if**
10: **end while**
11: **return** true

After line 1, how many pairs are there in the set S, assuming that there are n binary constraints?

Trace the execution of AC-3 algorithm

See Practice Question 1 in the notes on the course website.

Properties of the AC-3 Algorithm

- Does the order in which arcs are considered matter?
- Three possible outcomes of the arc consistency algorithm:
- Is AC-3 guaranteed to terminate?

Backtracking Search for 4-Queens Problem

Depth-first search v.s. Backtracking search on a CSP

Backtracking search is a special kind of Depth-first search, but they are not the same.

Why?

A CSP is commutative

- A CSP is commutative. Assigning values to variables in different orders will arrive at the same state.
- In each node, we should only consider one variable when generating successor states.

Backtracking Search

Algorithm 3 BACKTRACK(assignment, csp)

```
if assignment is complete then return true
var ← SELECT-UNASSIGNED-VARIABLE(csp)
for all value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if adding {var = value} satisfies every constraint then
        add {var = value} to assignment
        result ← BACKTRACK(assignment, csp)
        if result is true then return result
    end if
    remove {var = value} from assignment
end for
return false
```

Questions to consider

1. What inferences should we perform at each step of the search?
2. Which variable should we choose next?
   Which value of the variable should we try next?

Interleaving search and inferences

What inferences should be performed at each step in the search?

- Execute arc consistency algorithm before search.
- **Forward Checking**:
  - Simplified form of arc-consistency
  - Make every unassigned variable arc-consistent with the current variable.
- **Maintaining Arc Consistency (MAC)**:
  - Run the AC-3 algorithm

Backtracking with Inferences

Algorithm 4 BACKTRACK-INFERENCES(assignment, csp)

```
if assignment is complete then return true
var ← SELECT-UNASSIGNED-VARIABLE(csp)
for all value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if adding {var = value} satisfies every constraint then
        add {var = value} to assignment
        inf-result ← INFERENCES(assignment, csp)
        if inf-result is true then
            add the inference results to assignment
            result ← BACKTRACK(assignment, csp)
            if result is true then return result
        end if
    end if
    remove {var = value} and the inference results from assignment
end for
return false
```

Backtracking Search with Forward Checking

See Practice Question 2 in the notes on the course website.

CQ: Forward Checking

CQ: Consider the 4-queens problem with an empty assignment. Choose $x_0 = 0$. After this assignment, which of the following is the result of performing forward checking?

(A) $x_0 = 0, x_1 \in \{0, 1, 2, 3\}, x_2 \in \{0, 1, 2, 3\}, x_3 \in \{0, 1, 2, 3\}$

(B) $x_0 = 0, x_1 \in \{1, 2, 3\}, x_2 \in \{1, 2, 3\}, x_3 \in \{1, 2, 3\}$

(C) $x_0 = 0, x_1 \in \{2, 3\}, x_2 \in \{1, 3\}, x_3 \in \{1, 2\}$

(D) $x_0 = 0, x_1 \in \{3\}, x_2 \in \{1, 3\}, x_3 \in \{2\}$

CQ: Maintaining Arc Consistency

CQ: Consider the 4-queens problem with an empty assignment. Choose $x_0 = 0$. After this assignment, which of the following is the result of maintaining arc consistency?

(A) $x_0 = 0, x_1 \in \{2, 3\}, x_2 \in \{1, 3\}, x_3 \in \{1, 2\}$

(B) $x_0 = 0, x_1 \in \{3\}, x_2 \in \{1, 3\}, x_3 \in \{1\}$

(C) $x_0 = 0, x_1 \in \{3\}, x_2 \in \{1\}, x_3 \in \{1\}$

(D) $x_0 = 0, x_1 \in \{3\}, x_2 \in \{\}, x_3 \in \{1\}$

Which variable and value should we choose next?

Heuristics for selecting a variable

- **minimum-remaining-values** (MRV) heuristic: Choose the variable with the fewest values left in its domain.
- **degree** heuristic: Choose the variable that is involved in the largest number of constraints on other unassigned variables.
- When choosing a variable, apply the MRV heuristic first. Whenever there is a tie, use the degree heuristic to break ties.

Heuristics for selecting a value for a variable

- **least-constraining-value** heuristic: Select the value that **rules out the fewest values** for the neighbouring unassigned variables.

Backtracking with Inferences and Heuristics

Algorithm 5 BACKTRACK-INF-HEUR(assignment, csp)

```
if assignment is complete then return true
choose var based on MRV and DEGREE HEURISTICS
for all value in dom(var) chosen based on LCV HEURISTIC do
    if adding {var = value} satisfies every constraint then
        add {var = value} to assignment
        inf-result ← INFERENCES(assignment, csp)
        if inf-result is true then
            add the inference results to assignment
        end if
        result ← BACKTRACK(assignment, csp)
        if result is true then return result
    end if
    remove {var = value} and the inference results from assignment
end for
return false
```

CQ: Applying the least-constraining-value heuristic

CQ: Consider the following partial assignment for the 4-queens problem. (x_i denotes the row position of the queen in column i.)

$$x_0 = 0, x_1 \in \{2, 3\}, x_2 \in \{1, 3\}, x_3 \in \{1, 2\}$$

Based on the least-constraining-value heuristic, which value of x_1 should we choose?

(A) $x_1 = 2$

(B) $x_1 = 3$

Revisiting the Learning Goals

By the end of the lecture, you should be able to

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