Constraint Satisfaction Problems

Alice Gao

Lecture 4

Readings: RN 6.1 - 6.3. PM 4.1 - 4.4.

Outline

Learning Goals

By the end of the lecture, you should be able to

- Formulate a real-world problem as a constraint satisfaction problem.
- Trace the execution of the backtracking search algorithm.
- Verify whether a constraint is arc-consistent.
- Trace the execution of the AC-3 arc consistency algorithm.
- Trace the execution of the backtracking search algorithm with arc consistency.
- Trace the execution of the backtracking search algorithm with arc consistency and with heuristics for choosing variables and values.

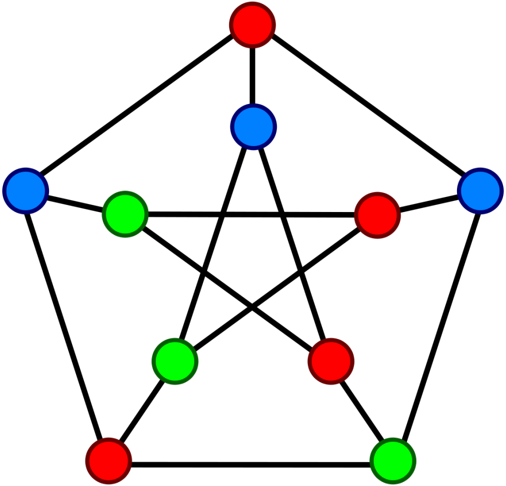
Evacuation Planning

- Instruct residents to follow a route at a given time.
- Two challenges:
  (1) deploy enough resources to give instructions.
  (2) ensure that the population comply with the instructions.
- Applied in a real-life case study and
  generated schedules close to the optimal ones from prior work.
- Even, C., Schutt, A., & Van Hentenryck, P. (2015).
  A constraint programming approach for non-preemptive evacuation scheduling.
  https://arxiv.org/pdf/1505.02487.pdf.

Crossword Puzzles

T	A	O	S		S	P	E	C	S		S	E	P	T
E	G	G	Y		A	L	A	A	P		I	L	I	A
T	H	R	E	E	P	E	N	N	Y		D	H	A	K
H	A	E		X	E	B	E	C		S	E	I	N	E
			V	O	L		D	H	O	W	S			
S	W	E	E	N	E	Y		A	N	A	T	M	A	N
P	R	Y	S		S	O	N		E	P	E	I	R	A
L	Y	R	I	C		N	O	S		S	P	A	E	R
I	L	I	C	E	S		R	U	C		P	O	C	K
T	Y	R	A	N	E	D		R	O	P	E	W	A	Y
			T	S	A	R	S		C	U	D			
B	A	G	I	E		O	I	N	K	S		T	S	K
O	L	E	O		S	W	E	E	P	S	T	A	K	E
Y	A	R	N		E	N	V	O	I		A	B	U	T
G	Y	M	S		I	D	E	N	T		G	U	G	A

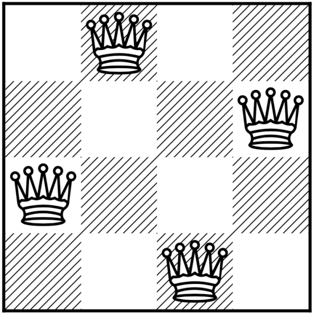
Graph Coloring Problem



Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

4-Queens Problem



Internal Structure of States

- Search algorithms are unaware of the internal structure of states.
- However, knowing a state's internal structure can help.

Defining a CSP

Each state contains

- A set X of variables: $\{X_1, X_2, ..., X_n\}$.
- A set D of domains: D_i is the domain for variable X_i, $\forall i$.
- A set C of constraints specifying allowable value combinations.

A solution is an assignment of values to all the variables that satisfy all the constraints.

4-Queens: State Definition in a CSP

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- Variables: x_0, x_1, x_2, x_3 where x_i is the row position of the queen in column i, where $i \in \{0, 1, 2, 3\}$.
  Assume that exactly one queen is in each column.

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  Assume that exactly one queen is in each column.

- Domains: $D_{x_i} = \{0, 1, 2, 3\}$ for all x_i.

- Constraints:

  No pair of queens are in the same row or diagonal.
  $(\forall i(\forall j((i \neq j) \rightarrow ((x_i \neq x_j) \wedge (|x_i - x_j| \neq |i - j|)))))$

  For example, $((x_0 \neq x_1) \wedge (|x_0 - x_1| \neq 1))$

4-Queens Incremental CSP Formulation

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- State: one queen per column in the leftmost k columns with no pair of queens attacking each other.
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- Initial state: the empty board, that is, _ _ _ _.

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- Initial state: the empty board, that is, _ _ _ _.
- Goal state: 4 queens on the board. No pair of queens are attacking each other. For example, 2 3 0 1 is a goal state.

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- Initial state: the empty board, that is, _ _ _ _.

- Goal state: 4 queens on the board. No pair of queens are attacking each other. For example, 2 3 0 1 is a goal state.

- Successor function: add a queen to the leftmost empty column such that it is not attacked by any other existing queen. For example, 0 _ _ _ has two successors 0 2 _ _ and 0 3 _ _.

4-Queens Complete Incremental CSP Formulation

- State: one queen per column in the leftmost k columns with no pair of queens attacking each other.
  - Variables: x_0, x_1, x_2, x_3 where x_i is the row position of the queen in column i, where $i \in \{0, 1, 2, 3, \_\}$. Exactly one queen is in each column. $x_i = \_$ denotes that column i does not have a queen.
  - Domains: $D_{x_i} = \{0, 1, 2, 3\}$ for all x_i.
  - Constraints: No pair of queens are in the same row or diagonal.
- Initial state: the empty board.
- Goal state: 4 queens on the board. No pair of queens are attacking each other.
- Successor function: add a queen to the leftmost empty column such that it is not attacked by any other existing queen.

Solve 4-Queens using Backtracking Search

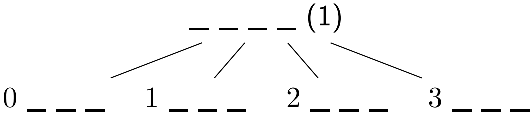
Step 0:

$\_\ \_\ \_\ \_$

	x_0	x_1	x_2	x_3
0				
1				
2				
3				

Solve 4-Queens using Backtracking Search

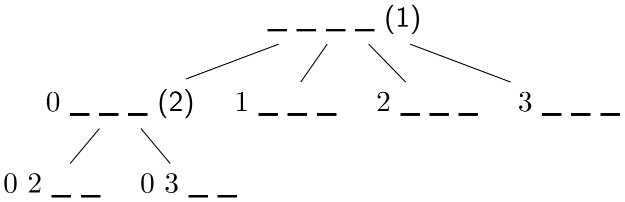
Step 1:



	x_0	x_1	x_2	x_3
0				
1				
2				
3				

Solve 4-Queens using Backtracking Search

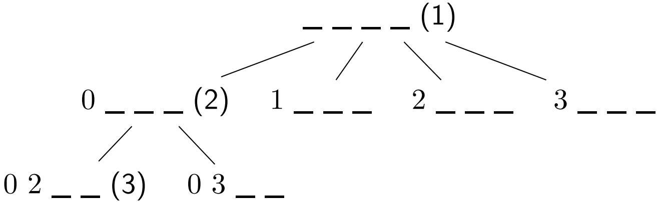
Step 2:



	x_0	x_1	x_2	x_3
0	Q	X	X	X
1	X	X		
2	X		X	
3	X			X

Solve 4-Queens using Backtracking Search

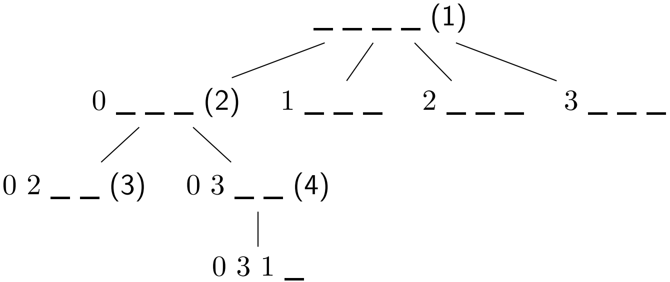
Step 3:



	x_0	x_1	x_2	x_3
0	Q	X	X	X
1	X	X	X	
2	X	Q	X	
3	X	X	X	X

Solve 4-Queens using Backtracking Search

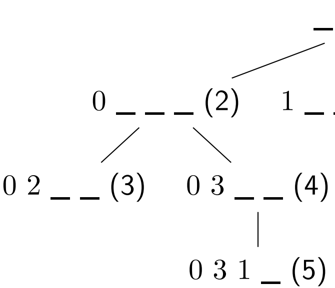
Step 4:



	x_0	x_1	x_2	x_3
0	Q	X	X	X
1	X	X		X
2	X	X	X	
3	X	Q	X	X

Solve 4-Queens using Backtracking Search

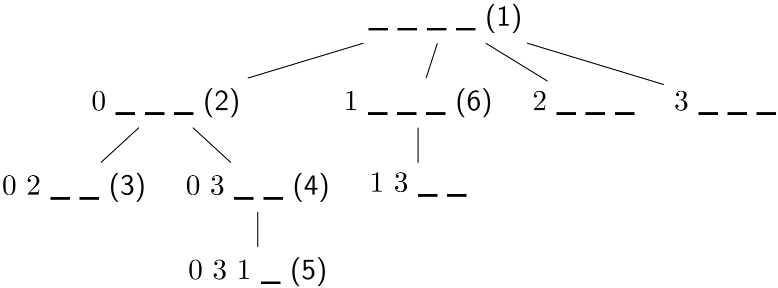
Step 5:



	x_0	x_1	x_2	x_3
0	Q	X	X	X
1	X	X	Q	X
2	X	X	X	X
3	X	Q	X	X

Solve 4-Queens using Backtracking Search

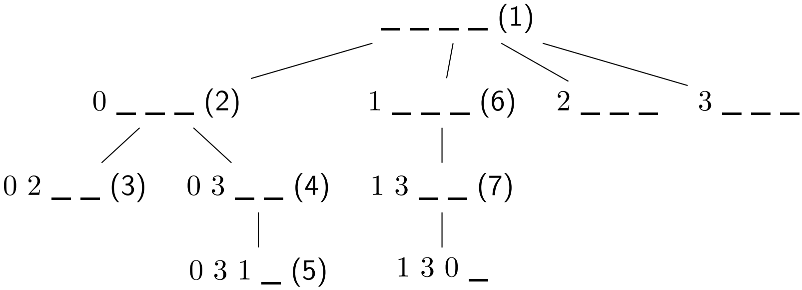
Step 6:



	x_0	x_1	x_2	x_3
0	X	X		
1	Q	X	X	X
2	X	X		
3	X		X	

Solve 4-Queens using Backtracking Search

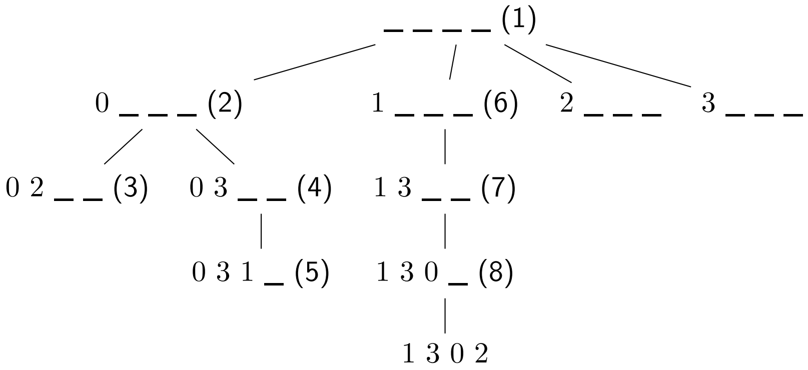
Step 7:



	x_0	x_1	x_2	x_3
0	X	X		
1	Q	X	X	X
2	X	X	X	
3	X	Q	X	X

Solve 4-Queens using Backtracking Search

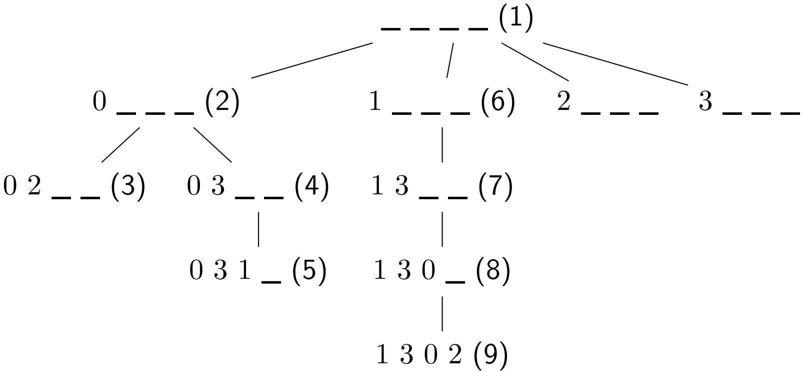
Step 8:



	x_0	x_1	x_2	x_3
0	X	X	Q	X
1	Q	X	X	X
2	X	X	X	
3	X	Q	X	X

Solve 4-Queens using Backtracking Search

Step 9:



	x_0	x_1	x_2	x_3
0	X	X	Q	X
1	Q	X	X	X
2	X	X	X	Q
3	X	Q	X	X

The Idea of Arc Consistency

$x_0 = 0$ and $x_1 = 2$ do not lead to a solution. Why?

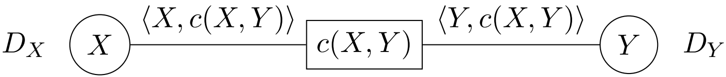
Q			
	Q		

Handling Different Types of Constraints

- Consider binary constraints only.
- How should we handle unary constraints?
- How should be handle constraints involving 3 or more variables?

Notation for an Arc

- X and Y are two variables. $c(X,Y)$ is a binary constraint.



- $\langle X, c(X,Y)\rangle$ denotes an arc.
  X is the primary variable and Y is the secondary variable.

The Arc Consistency Definition

Definition (Arc Consistency)

An arc $\langle X, c(X,Y) \rangle$ is arc-consistent if and only if for every value v in D_X, there is a value w in D_Y such that (v,w) satisfies the constraint $c(X,Y)$.

Applying The Arc Consistency Definition

Question: Consider the constraint $X < Y$. Let $D_X = \{1, 2\}$ and $D_Y = \{1, 2\}$. Is the arc $\langle X, X < Y \rangle$ consistent?

(A) Yes.

(B) No.

(C) I don't know.

The AC-3 Arc Consistency Algorithm

Algorithm 1 The AC-3 Algorithm

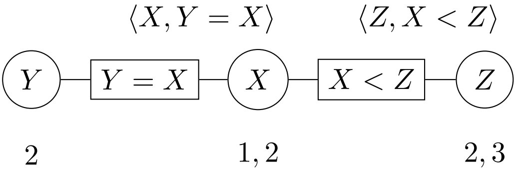
put every arc in the set S.
while S is not empty **do**
  select and remove $\langle X, c(X,Y) \rangle$ from S
  remove every value in D_X that doesn't have a value in D_Y that satisfies the constraint $c(X,Y)$
  if D_X was reduced **then**
    if D_X is empty **then return** false
    for every $Z \neq Y$, add $\langle Z, c'(Z,X) \rangle$ to S
return true

Why do we need to add arcs back to S?

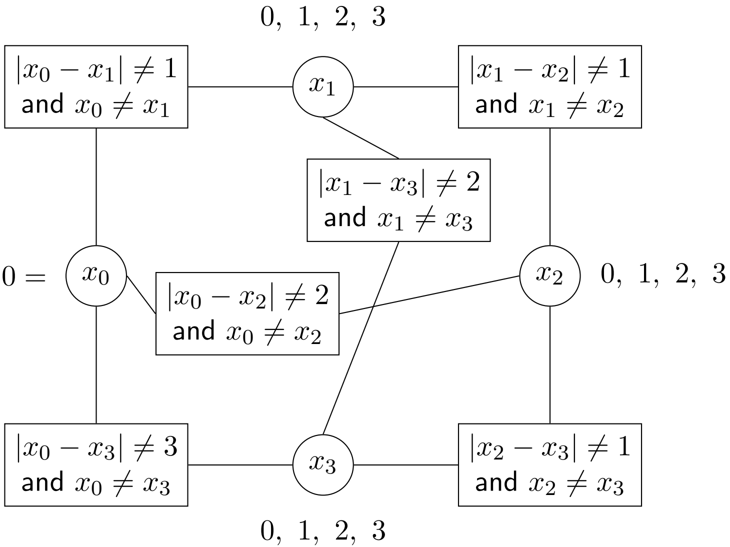
Q: After reducing a variable's domain,
we may need to add arcs back to S. Why?

A: Reducing a variable's domain may
cause a previously consistent arc to become inconsistent.

Example:



Trace the AC-3 Algorithm on 4-Queens Problem



Properties of the AC-3 Algorithm

- Does the order of removing arcs matter?
- Three possible outcomes of the arc consistency algorithm:
- Is AC-3 guaranteed to terminate?

Combining Backtracking and Arc Consistency

1. Perform backtracking search.
2. After each value assignment, perform arc consistency.
3. If a domain is empty, terminate and return no solution.
4. If a unique solution is found, return the solution.
5. Otherwise, continue with backtracking search
   on the unassigned variables.

Solving 4-Queens Problem with Backtracking and AC-3

Revisiting the Learning Goals

By the end of the lecture, you should be able to

- Formulate a real-world problem as a constraint satisfaction problem.
- Trace the execution of the backtracking search algorithm.
- Verify whether a constraint is arc-consistent.
- Trace the execution of the AC-3 arc consistency algorithm.
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