Home or dancing - Friends who enjoy each other's company

Alice and Anna are best friends in grad school. They both enjoy each other's company, but neither can communicate with the other before deciding whether to stay at home (where they would not see each other) or go swing dancing this evening (where they could see each other). Each prefers going dancing to being at home. This game can be represented by the following payoff matrix.

		Anna	
		home	*dancing*
Alice	*home*	$(0,0)$	$(0,1)$
	dancing	$(1,0)$	$(2,2)$

A normal form game consists of:

- A set of player I. $I = \{Alice, Anna\}$.
- Each player $i \in I$ has a set of actions A_i. $A_{Alice} = A_{Anna} = \{home, dancing\}$.
- A payoff matrix. Once each player chooses an action, we have an outcome of the game. For example, $(home, dancing)$ is an outcome. Each agent has a utility for each outcome. For the outcome $(home, dancing)$, the utility pair $(0, 1)$ means that Alice has a utility of 0 and Anna has a utility of 1 for this outcome.

Players choose their actions

- at the same time.
- without communicating with each other.
- without knowing other players' actions.

Each player chooses a strategy, which can be pure or mixed.

- A mixed strategy is a probability distribution over all the actions.
- A pure strategy is an action. (A pure strategy is a special type of mixed strategies where one action is played with probability 1.)

A strategy profile σ contains a strategy σ_i for each player i.

For this lecture, we will focus on pure strategies, which are actions.

Pure strategy profiles for this game: (home, home), (home, dancing), (dancing, home), (dancing, dancing).

For a strategy profile σ, let σ_i be the strategy of agent i and let σ_{-i} denote the strategies of all agents except i. $\sigma_{-i} = \{\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n\}$.

Let $U_i(\sigma) = U_i(\sigma_i, \sigma_{-i})$ denote the utility of agent i under the strategy profile σ.

What would Alice and Anna do?

Dominance and dominant strategy equilibrium

- For player i, **a strategy** σ_i **dominates strategy** σ_i' iff
  - $U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i}$, and
  - $U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i', \sigma_{-i}), \exists \sigma_{-i}$
- **A dominant strategy** dominates all other strategies.
- When each player has a dominant strategy, the combination of those strategies is called **a dominant strategy equilibrium**.

(CQ) For Alice, does going dancing dominate staying at home? Does staying at home dominate going dancing?

For Alice, going dancing is strictly better than staying at home, regardless of what Anna does. Thus, going dancing is a dominant strategy for Alice. This is the same for Anna.

(CQ) How many of the four outcomes are dominant strategy equilibria?

Alice's dominant strategy is dancing. Anna's dominant strategy is dancing as well. Thus, the only dominant strategy equilibrium is (dancing, dancing).

Alice and Anna do not need to communicate beforehand. Each pursue their own interest and the best outcome occurs for both.

Signing up for the same activity

Alice and Anna would like to sign up for an activity together. They both prefer dancing over running. They also prefer signing up for the same activity over signing up for two different activities.

		Anna *dancing*	Anna *running*
Alice	*dancing*	$(2, 2)$	$(0, 0)$
Alice	*running*	$(0, 0)$	$(1, 1)$

Which outcomes are dominant strategy equilibria of this game?

- (CQ) For Alice, does dancing dominate running? No.

  For Alice, does running dominate dancing? No.

  Depending on what Anna's action is, Alice prefers different actions. Thus, Alice does not have a dominant strategy. Neither does Anna.

- (CQ) There is no dominant strategy equilibrium since each player does not have a dominant strategy.

Best Response and Nash equilibrium

- **Best response:** Given a strategy profile (σ_i, σ_{-i}), agent i's strategy σ_i is a best response to the other agents' strategies σ_{-i} if and only if

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \neq \sigma_i.$$

  A rational player always plays the best response to all other players' strategies.

- **Nash equilibrium:** A strategy profile σ is a Nash equilibrium if and only if each agent i's strategy σ_i is a best response to the other agents' strategies σ_{-i}.

  Nash equilibrium: every agent is choosing the best strategy given the strategies of all other agents.

  Not a Nash equilibrium: at least one agent has a better strategy than their current strategy given other agents' strategies.

Which outcomes are Nash equilibria of this game?

- (CQ) Best response:

  Given the strategy profile (*dancing*, *running*), is Alice's action a best response to Anna's action? No. Alice prefers running if Anna goes running. Is Anna's action a best response to Alice's action? No. Anna prefers dancing if Alice goes dancing.

- (CQ) Nash equilibria

  There are two Nash equilibria of this game: (dancing, dancing), and (running, running). At each Nash equilibrium, neither Alice nor Anna wants to change her action.

As far as Nash equilibrium is concerned, both outcomes (dancing, dancing) and (running, running) are equilibria and might be played. However, our intuition tells us that (dancing, dancing) is better for both players than (running, running). This intuition is not captured by the concept of Nash equilibrium.

How do we capture this intuition? Which Nash equilibrium will the players choose?

Pareto dominance and optimality:

- **Pareto dominance:** An outcome o Pareto dominates another outcome o' iff every player is weakly better off in o and at least one player is strictly better off in o.

- **A Pareto optimal outcome:** An outcome o is Pareto optimal iff no other outcome o' Pareto dominates o.

  Notice that this definition is weaker than claiming that a Pareto optimal outcome must Pareto dominate all other outcomes. It only says that a Pareto optimal outcome cannot be Pareto dominated by any other outcome.

(CQ) Which of the four outcomes are Pareto optimal?

It is easy to see several Pareto dominance relationships: (*dancing*, *dancing*) Pareto dominates all other outcomes. (*running*, *running*) Pareto dominates both outcomes where the two players miscoordinate ((*running*, *dancing* and (*dancing*, *running*)). Given these Pareto dominance relationships, the only outcome that is not Pareto dominated by any other outcome is (*dancing*, *dancing*).

Thus, (*dancing*, *dancing*) is the only Pareto optimal outcome.

Prisoner's dilemma

Alice and Anna have been caught by the police. Each has been offered a deal to testify against the other. They had originally agreed not to testify against each other. However, since this agreement cannot be enforced, each must choose whether to honour it. If both refuse to testify, both will be convicted of a minor charge due to lack of evidence and serve 1 year in prison. If only one testifies, the defector will go free and the other one will be convicted of a serious charge and serve 3 years in prison. If both testify, both will be convicted of a major charge and serve 2 years in prison.

		Anna	
		refuse	testify
Alice	refuse	$(-1, -1)$	$(-3, 0)$
	testify	$(0, -3)$	$(-2, -2)$