Given the Bayesian network for the Holmes scenario (included on the last page), we would like to answer the following question:

What is the probability that a burglary is happening given that Dr. Watson and Mrs. Gibbon both call? In other words, what is $P(B|W \wedge G)$?

The variable elimination algorithm

We will calculate the following conditional probability distribution using the variable elimination algorithm.

$$\begin{bmatrix} P(B|W \wedge G) \\ P(\neg B|W \wedge G) \end{bmatrix}$$

In other words, we want to calculate the following expression:

$$\sum_{e \in \{t,f\}} \sum_{a \in \{t,f\}} \sum_{r \in \{t,f\}} P(E=e)P(B=b)P(R=r|E=e)P(A=a|B=b \wedge E=e) \\ P(W=t|A=a)P(G=t|A=a)$$

Step 1: Given a Bayesian network, construct a factor for each conditional probability distribution.

Define factor $f_1(E)$ to correspond to $P(E=e)$.

$$f_1(E) = \begin{bmatrix} P(E) \\ P(\neg E) \end{bmatrix} = \begin{bmatrix} 0.0003 \\ 0.9997 \end{bmatrix}$$

Define factor $f_2(B)$ to correspond to $P(B=b)$.

$$f_2(B) = \begin{bmatrix} P(B) \\ P(\neg B) \end{bmatrix} = \begin{bmatrix} 0.0001 \\ 0.9999 \end{bmatrix}$$

Define factor $f_3(R, E)$ to correspond to $P(R=r|E=e)$.

$$f_3(R, E) = \begin{bmatrix} P(R|E) & P(\neg R|E) \\ P(R|\neg E) & P(\neg R|\neg E) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.0002 & 0.9998 \end{bmatrix}$$

Define factor $f_4(A, B, E)$ to correspond to $P(A=a|B=b \wedge E=e)$.

$$f_4(A, B=t, E) = \begin{bmatrix} P(A|B \wedge E) & P(\neg A|B \wedge E) \\ P(A|B \wedge \neg E) & P(\neg A|B \wedge \neg E) \end{bmatrix} = \begin{bmatrix} 0.96 & 0.04 \\ 0.95 & 0.05 \end{bmatrix}$$

$$f_4(A, B = f, E) = \begin{bmatrix} P(A|\neg B \wedge E) & P(\neg A|\neg B \wedge E) \\ P(A|\neg B \wedge \neg E) & P(\neg A|\neg B \wedge \neg E) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.01 & 0.99 \end{bmatrix}$$

Define factor $f_5(W, A)$ to correspond to $P(W = w|A = a)$.

$$f_5(W, A) = \begin{bmatrix} P(W|A) & P(\neg W|A) \\ P(W|\neg A) & P(\neg W|\neg A) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

Define factor $f_6(G, A)$ to correspond to $P(G = g|A = a)$.

$$f_6(G, A) = \begin{bmatrix} P(G|A) & P(\neg G|A) \\ P(G|\neg A) & P(\neg G|\neg A) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.04 & 0.96 \end{bmatrix}$$

Our expression becomes:

$$\sum_{e\in\{t,f\}} \sum_{a\in\{t,f\}} \sum_{r\in\{t,f\}} f_1(E) \times f_2(B) \times f_3(R, E) \times f_4(A, B, E) \times f_5(W, A) \times f_6(G, A)$$

Step 2: Restrict the factors. Set the observed variables to their observed values.

In the original query, we have observed that W and G are both true. We will modify the corresponding factors to reflect this.

Define factor $f_7(A) = f_5(W = t, A)$.

$$f_7(A) = \begin{bmatrix} P(W|A) \\ P(W|\neg A) \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix}$$

Define factor $f_8(A) = f_6(G = t, A)$.

$$f_8(A) = \begin{bmatrix} P(G|A) \\ P(G|\neg A) \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.04 \end{bmatrix}$$

Our expression becomes:

$$\sum_{e\in\{t,f\}} \sum_{a\in\{t,f\}} \sum_{r\in\{t,f\}} f_1(E) \times f_2(B) \times f_3(R, E) \times f_4(A, B, E) \times f_7(A) \times f_8(A)$$

Step 3: Sum out each hidden variable according to an order.

There are three hidden variables, A, E, and R. We will sum them out in the order of R, E, and A.

First, we will sum out R.

- The factors containing R are: $f_3(R, E)$.

  This is equivalent to re-arranging the equation, as follows.

$$\sum_{e \in \{t,f\}} \sum_{a \in \{t,f\}} f_1(E) \times f_2(B) \times f_4(A, B, E) \times f_7(A) \times f_8(A) \times \sum_{r \in \{t,f\}} f_3(R, E)$$

- Sum out R from $f_3(R, E)$.

  Define a new factor $f_9(E) = \sum_r f_3(R, E)$.

  f_9 is

$$f_9(E) = \begin{bmatrix} f_9(E) \\ f_9(\neg E) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

  Our expression becomes

$$\sum_{e \in \{t,f\}} \sum_{a \in \{t,f\}} f_1(E) \times f_2(B) \times f_4(A, B, E) \times f_7(A) \times f_8(A) \times f_9(E)$$

Next, we will sum out E.

- The factors containing E are: $f_1(E)$, $f_4(A, B, E)$, and $f_9(E)$.

  This is equivalent to re-arranging the equation, as follows.

$$\sum_{a \in \{t,f\}} f_2(B) \times f_7(A) \times f_8(A) \times \sum_{e \in \{t,f\}} f_1(E) \times f_4(A, B, E) \times f_9(E)$$

- Multiply the factors together. This is a pointwise multiplication.

  Let $f_{10}(A, B, E) = f_1(E) \times f_4(A, B, E) \times f_9(E)$ where $\times$ is pointwise multiplication.

  f_{10} is

$$f_{10}(A, B = t, E) = \begin{bmatrix} f_{10}(A = t, B = t, E = t) & f_{10}(A = f, B = t, E = t) \\ f_{10}(A = t, B = t, E = f) & f_{10}(A = f, B = t, E = f) \end{bmatrix} = \begin{bmatrix} 0.000288 & 0.000012 \\ 0.949715 & 0.049985 \end{bmatrix}$$

$$f_{10}(A, B = f, E) = \begin{bmatrix} f_{10}(A = t, B = f, E = t) & f_{10}(A = f, B = f, E = t) \\ f_{10}(A = t, B = f, E = f) & f_{10}(A = f, B = f, E = f) \end{bmatrix} = \begin{bmatrix} 0.00006 & 0.00024 \\ 0.009997 & 0.989703 \end{bmatrix}$$

  Our expression becomes:

$$\sum_{a \in \{t,f\}} f_2(B) \times f_7(A) \times f_8(A) \times \sum_{e \in \{t,f\}} f_{10}(A, B, E)$$

- Sum out E from $f_{10}(A, B, E)$.

  Define a new factor $f_{11}(A, B) = \sum_r f_{10}(A, B, E)$.

  f_{11} is

$$f_{11}(A, B) = \begin{bmatrix} f_8(A = t, B = t) & f_8(A = t, B = f) \\ f_8(A = f, B = t) & f_8(A = f, B = f) \end{bmatrix} = \begin{bmatrix} 0.950003 & 0.010057 \\ 0.049997 & 0.989943 \end{bmatrix}$$

  Our expression becomes

$$\sum_{a \in \{t,f\}} f_2(B) \times f_7(A) \times f_8(A) \times f_{11}(A, B)$$

Finally, we will sum out A.

- The factors containing A are: $f_7(A)$, $f_8(A)$, and $f_{11}(A, B)$.

  This is equivalent to re-arranging the equation, as follows.

$$f_2(B) \times \sum_{a \in \{t,f\}} f_7(A) \times f_8(A) \times f_{11}(A, B)$$

- Multiply the factors together. This is a pointwise multiplication.

  Let $f_{12}(A, B) = f_7(A) \times f_8(A) \times f_{11}(A, B)$.

  $f_{12}(A, B)$ is

$$f_{12}(A, B) = \begin{bmatrix} f_{12}(A = t, B = t) & f_{12}(A = t, B = f) \\ f_{12}(A = f, B = t) & f_{12}(A = f, B = f) \end{bmatrix} = \begin{bmatrix} 0.30400096 & 0.00321824 \\ 0.00079995 & 0.015839088 \end{bmatrix}$$

  Our expression becomes:

$$f_2(B) \times \sum_{a \in \{t,f\}} f_{12}(A, B)$$

- Sum out A from $f_{12}(A, B)$.

  Define a new factor $f_{13}(A, B) = \sum_r f_{12}(A, B)$.

  $f_{13}(B)$ is

$$f_{13}(B) = \begin{bmatrix} f_{13}(B = t) \\ f_{13}(B = f) \end{bmatrix} = \begin{bmatrix} 0.30480091 \\ 0.01905733 \end{bmatrix}$$

  Our expression becomes

$$f_2(B) \times f_{13}(B)$$

Step 4: Multiple the remaining factors.

If there are more than one factor left, multiply the factors together.

Let $f_{14}(B) = f_2(B) \times f_{13}(B)$.

$f_{14}(B)$ is

$$f_{14}(B) = \begin{bmatrix} f_{14}(B = t) \\ f_{14}(B = f) \end{bmatrix} = \begin{bmatrix} 0.000030480091 \\ 0.0190554223 \end{bmatrix}$$

This factor corresponds to the following probabilities:

$$\begin{bmatrix} P(B \wedge W \wedge G) \\ P(\neg B \wedge W \wedge G) \end{bmatrix}$$

Step 5: Normalize the resulting factor.

The final result we need is a conditional probability distribution where all the probabilities should sum to one. Therefore, we will normalize the factor so that it represents a probability distribution.

Let $f_{15}(B) = \dfrac{f_{14}(B)}{\sum_b f_{14}(B = b)}$.

$f_{15}(B)$ is

$$f_{15}(B) = \begin{bmatrix} f_{15}(B = t) \\ f_{15}(B = f) \end{bmatrix} = \begin{bmatrix} 0.001597 \\ 0.998403 \end{bmatrix}$$

This factor corresponds to the following probabilities:

$$\begin{bmatrix} P(B|W \wedge G) \\ P(\neg B|W \wedge G) \end{bmatrix}$$

The Holmes scenario

Mr. Holmes lives in a high crime area and therefore has installed a burglar alarm. He relies on his neighbors to phone him when they hear the alarm sound. Mr. Holmes has two neighbors, Dr. Watson and Mrs. Gibbon.

Unfortunately, his neighbors are not entirely reliable. Dr. Watson is known to be a tasteless practical joker and Mrs. Gibbon, while more reliable in general, has occasional drinking problems.

Mr. Holmes also knows from reading the instruction manual of his alarm system that the device is sensitive to earthquakes and can be triggered by one accidentally. He realizes that if an earthquake has occurred, it would surely be on the radio news.