The Holmes scenario

Mr. Holmes lives in a high crime area and therefore has installed a burglar alarm. He relies on his neighbors to phone him when they hear the alarm sound. Mr. Holmes has two neighbors, Dr. Watson and Mrs. Gibbon.

Unfortunately, his neighbors are not entirely reliable. Dr. Watson is known to be a tasteless practical joker and Mrs. Gibbon, while more reliable in general, has occasional drinking problems.

Mr. Holmes also knows from reading the instruction manual of his alarm system that the device is sensitive to earthquakes and can be triggered by one accidentally. He realizes that if an earthquake has occurred, it would surely be on the radio news.

Inferences using the joint distribution

Here is a joint distribution of the three random variables Alarm, Watson and Gibbon.

A			$\neg A$		
	G	$\neg G$		G	$\neg G$
W	0.032	0.048	W	0.036	0.324
$\neg W$	0.008	0.012	$\neg W$	0.054	0.486

1. What is probability that **the alarm is NOT going and Dr. Watson is calling**?

$$\begin{aligned}
&P(\neg A \land W) \\
&= P(\neg A \land W \land G) + P(\neg A \land W \land \neg G) \\
&= 0.036 + 0.324 \\
&= 0.36
\end{aligned}$$

2. What is probability that **the alarm is going and Mrs. Gibbon is NOT calling**?

$$\begin{aligned}
&P(A \land \neg G) \\
&= P(A \land W \land \neg G) + P(A \land \neg W \land \neg G) \\
&= 0.048 + 0.012 \\
&= 0.06
\end{aligned}$$

3. What is the probability that **the alarm is NOT going**?

$$
\begin{aligned}
&P(\neg A) \\
&= P(\neg A \wedge W \wedge G) + P(\neg A \wedge \neg W \wedge G) + P(\neg A \wedge negW \wedge \neg G) + P(\neg A \wedge W \wedge \neg G) \\
&= 0.036 + 0.054 + 0.486 + 0.324 \\
&= 0.9
\end{aligned}
$$

4. What is probability that **Dr. Watson is calling given that the alarm is NOT going**?

$$
\begin{aligned}
&P(W|\neg A) \\
&= P(W \wedge \neg A)/P(\neg A) \\
&= 0.36/0.9 = 0.4
\end{aligned}
$$

5. What is probability that **Mrs. Gibbon is NOT calling given that the alarm is going**?

$$
\begin{aligned}
&P(\neg G \wedge A) \\
&= P(A \wedge W \wedge \neg G) + P(A \wedge \neg W \wedge \neg G) \\
&= 0.048 + 0.012 \\
&= 0.06 \\
&P(\neg G|A) \\
&= P(\neg G \wedge A)/P(A) \\
&= 0.06/0.1 \\
&= 0.6
\end{aligned}
$$

Inferences using the prior and conditional probabilities

The prior probabilities:

$P(A) = 0.1$ $\quad$ $P(W) = 0.45$ $\quad$ $P(G) = 0.12$

The conditional probabilities

$P(W|A) = 0.9$	$P(W|\neg A) = 0.4$	$P(G|A) = 0.3$	$P(G|\neg A) = 0.1$
$P(W|A \wedge G) = 0.9$	$P(W|\neg A \wedge G) = 0.4$	$P(G|A \wedge W) = 0.3$	$P(G|\neg A \wedge W) = 0.1$
$P(W|A \wedge \neg G) = 0.9$	$P(W|\neg A \wedge \neg G) = 0.4$	$P(G|A \wedge \neg W) = 0.3$	$P(G|\neg A \wedge \neg W) = 0.1$

1. What is probability that **the alarm is going, Dr. Watson is calling and Mrs. Gibbon is NOT calling**?

$$P(\neg G|A \wedge W) = 1 - P(G|A \wedge W) = 1 - 0.3 = 0.7$$

$$\begin{aligned} & P(A \wedge W \wedge \neg G) \\ & = P(A) * P(W|A) * P(\neg G|A \wedge W) \\ & = 0.1 * 0.9 * 0.7 \\ & = 0.063 \end{aligned}$$

2. What is probability that **the alarm is NOT going, Dr. Watson is NOT calling and Mrs. Gibbon is NOT calling**?

$$\begin{aligned} & P(\neg A \wedge \neg W \wedge \neg G) \\ & = P(\neg A) * P(\neg W|\neg A) * P(\neg G|\neg A \wedge \neg W) \\ & = 0.9 * 0.6 * 0.9 \\ & = 0.486 \end{aligned}$$

3. What is the probability that **the alarm is NOT going given that Dr. Watson is calling**?

$$\begin{aligned} & P(\neg A|W) = \frac{P(\neg A \wedge W)}{P(W)} \\ & = \frac{P(\neg A)P(W|\neg A)}{P(W)} \\ & = \frac{P(\neg A)P(W|\neg A)}{P(\neg A)P(W|\neg A) + P(A)P(W|A)} \\ & = \frac{0.9 * 0.4}{0.9 * 0.4 + 0.1 * 0.9} = 0.36/0.45 = 0.8 \end{aligned}$$

4. What is the probability that **the alarm is going given that Mrs. Gibbon is NOT calling**?

$$
\begin{aligned}
&P(\neg G|A) = 1 - P(G|A) = 1 - 0.3 = 0.7 \\
&P(A|\neg G) \\
&= \frac{P(A \wedge \neg G)}{P(\neg G)} \\
&= \frac{P(A)P(\neg G|A)}{P(A)P(\neg G|A) + P(\neg A)P(\neg G|\neg A)} \\
&= \frac{0.1 * 0.7}{0.1 * 0.7 + 0.9 * 0.9} = 0.07/0.88 = 0.080
\end{aligned}
$$

Unconditional and Conditional Independence

Example 1: Burglary, Alarm and Watson

$P(B) = 0.1$ $\quad$ $P(A|B) = 0.9$ $\quad$ $P(A|\neg B) = 0.1$

$P(W|B \wedge A) = 0.8$ $\quad$ $P(W|\neg B \wedge A) = 0.8$ $\quad$ $P(W|B \wedge \neg A) = 0.4$ $\quad$ $P(W|\neg B \wedge \neg A) = 0.4$

1. Is Burglary independent of Watson?

   No. Burglary is not independent of Watson. The probability of a Burglary affects the probability of the alarm going off, which in turn affects the probability of Watson calling. If the probability of a Burglary increases, then the probability of the alarm going off increases, and the probability of Watson calling will increase as well. If the probability of Watson calling increases, it must be that the probability that the alarm going off has increased, which means that the probability of a Burglary must have increased as well. Since the probabilities of Burglary and Watson affect each other, they are not independent.

   To show this formally, it is sufficient to show that $P(B) \neq P(B|W)$.

$$
\begin{aligned}
&P(B) = 0.1 \\
&P(B \wedge W) \\
&= P(B \wedge A \wedge W) + P(B \wedge \neg A \wedge W) \\
&= P(B)P(A|B)P(W|A \wedge B) + P(B)P(\neg A|B)P(W|\neg A \wedge B) \\
&= 0.1 * 0.9 * 0.8 + 0.1 * 0.1 * 0.4 \\
&= 0.076 \\
&P(\neg B \wedge W) \\
&= P(\neg B)P(A|\neg B)P(W|A \wedge \neg B) + P(\neg B)P(\neg A|\neg B)P(W|\neg A \wedge \neg B) \\
&= 0.9 * 0.1 * 0.8 + 0.9 * 0.9 * 0.4 \\
&= 0.396 \\
&P(W) \\
&= P(B \wedge W) + P(\neg B \wedge W) \\
&= 0.076 + 0.396 = 0.472 \\
&P(B|W) = P(B \wedge W)/P(W) \\
&= 0.076/0.472 \approx 0.161
\end{aligned}
$$

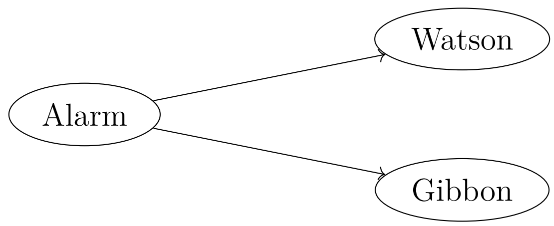
2. Is Burglary conditionally independent of Watson given Alarm?

   Yes. Burglary and Watson could only affect each other through Alarm. If we know whether the alarm is going off or not, then Burglary and Watson cannot affect each other in any way.

   To prove this mathematically, we need to show that all the following equations hold.

$$
\begin{aligned}
&P(B|A \land W) = P(B|A \land \neg W) = P(B|A) \\
&P(\neg B|A \land W) = P(\neg B|A \land \neg W) = P(\neg B|A) \\
&P(B|\neg A \land W) = P(B|\neg A \land \neg W) = P(B|\neg A) \\
&P(\neg B|\neg A \land W) = P(\neg B|\neg A \land \neg W) = P(\neg B|\neg A)
\end{aligned}
$$

Example 2: Alarm, Watson, and Gibbon



$P(A) = 0.1$ $\quad P(W|A) = 0.8$ $\quad P(W|\neg A) = 0.4$

$P(G|W \wedge A) = 0.4$ $\quad P(G|\neg W \wedge A) = 0.4$ $\quad P(G|W \wedge \neg A) = 0.1$ $\quad P(G|\neg W \wedge \neg A) = 0.1$

1. Is Watson independent of Gibbon?

   No. Watson is not independent of Gibbon. If Watson is more likely to call, then this must mean that the alarm is more likely to go off, which means that Gibbon is more likely to call as well. Therefore, changing the probability of Watson calling also changes the probability of Gibbon calling. Watson and Gibbon affect each other because they are both caused by the alarm going off.

   To show this formally, we can show that $P(G) \neq P(G|W)$, as follows.

$$
\begin{aligned}
&P(G \wedge W)\\
&= P(A \wedge W \wedge G) + P(\neg A \wedge W \wedge G)\\
&= 0.1 * 0.8 * 0.4 + 0.9 * 0.4 * 0.1 = 0.068\\
&P(G \wedge \neg W)\\
&= P(A \wedge \neg W \wedge G) + P(\neg A \wedge \neg W \wedge G)\\
&= 0.1 * 0.2 * 0.4 + 0.9 * 0.6 * 0.1 = 0.062\\
&P(G)\\
&= P(G \wedge W) + P(G \wedge \neg W)\\
&= 0.068 + 0.062 = 0.13\\
&P(W) = P(A)P(W|A) + P(\neg A)P(W|\neg A)\\
&= 0.1 * 0.8 + 0.9 * 0.4 = 0.44\\
&P(G|W)\\
&= P(G \wedge W)/P(W)\\
&= 0.068/0.44 \approx 0.155
\end{aligned}
$$

   You can see that, if we know that Watson is calling, then we believe that the probability of Gibbon calling increased from 0.13 to 0.155.

2. Is Watson independent of Gibbon given Alarm?

   Yes, Watson and Gibbon are conditionally independent given Alarm. The only way for Watson and Gibbon to affect each other is through Alarm. If we know whether the alarm is going off or not, then knowing whether Watson is calling does not affect our belief of whether Gibbon is calling.

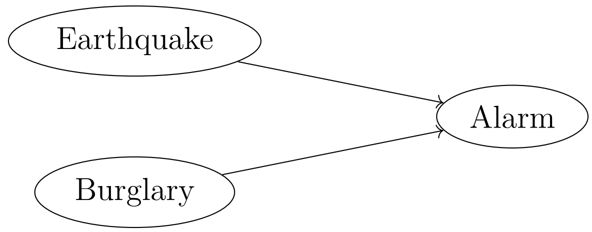
   To prove this formally, we need to verify all the following equations.

$$
\begin{aligned}
&P(W|G \wedge A) = P(W|\neg G \wedge A) = P(W|A) \\
&P(\neg W|G \wedge A) = P(\neg W|\neg G \wedge A) = P(\neg W|A) \\
&P(W|G \wedge \neg A) = P(W|\neg G \wedge \neg A) = P(W|\neg A) \\
&P(\neg W|G \wedge \neg A) = P(\neg W|\neg G \wedge \neg A) = P(\neg W|\neg A)
\end{aligned}
$$

   This is one calculation.

$$
\begin{aligned}
&P(G|W \wedge A) = 0.4 \\
&P(G \wedge A) \\
&= P(G \wedge W \wedge A) + P(G \wedge \neg W \wedge A) \\
&= 0.1 * 0.8 * 0.4 + 0.1 * 0.2 * 0.4 = 0.04 \\
&P(G|A) \\
&= P(G \wedge A)/P(A) \\
&= 0.04/0.1 = 0.4
\end{aligned}
$$

Example 3: Earthquake, Burglary and Alarm



$P(E) = 0.1$ $\quad P(B|E) = 0.2$ $\quad P(B|\neg E) = 0.2$

$P(A|B \land E) = 0.9$ $\quad P(A|\neg B \land E) = 0.2$ $\quad P(A|B \land \neg E) = 0.8$ $\quad P(A|\neg B \land \neg E) = 0.1$

1. Is Earthquake independent of Burglary?

   Yes. Earthquake is independent of Burglary (assuming that looting is not more common during an earthquake.)

   To show this formally, we need to verify all of the following equations:

$$P(E|B) = P(E|\neg B) = P(E)$$
$$P(\neg E|B) = P(\neg E|\neg B) = P(\neg E)$$

2. Is Earthquake conditionally independent of Burglary given Alarm?

   No, Earthquake is not conditionally independent of Burglary given Alarm. If an earthquake is happening, then it is less likely that the alarm going off is caused by a burglary.

$$P(E|B \land A) \neq P(E|\neg B \land A)$$

$$\begin{aligned}
&P(E \land B \land A) \\
&= P(E)P(B|E)P(A|B \land E) \\
&= 0.1 * 0.2 * 0.9 = 0.018 \\
&P(\neg E \land B \land A) \\
&= 0.9 * 0.2 * 0.8 = 0.144 \\
&P(B \land A) \\
&= P(E \land B \land A) + P(\neg E \land B \land A) \\
&= 0.018 + 0.144 = 0.162 \\
&P(E|B \land A) = P(E \land B \land A)/P(B \land A) = 0.018/0.162 \approx 0.11
\end{aligned}$$

$$\begin{aligned}
&P(E \land \neg B \land A) \\
&= P(E)P(\neg B|E)P(A|\neg B \land E) \\
&= 0.1 * 0.8 * 0.2 = 0.016 \\
&P(\neg E \land \neg B \land A) \\
&= 0.9 * 0.8 * 0.1 = 0.072 \\
&P(\neg B \land A) \\
&= P(E \land \neg B \land A) + P(\neg E \land \neg B \land A) \\
&= 0.016 + 0.072 = 0.088 \\
&P(E|\neg B \land A) = P(E \land \neg B \land A)/P(B \land A) = 0.016/0.088 \approx 0.182
\end{aligned}$$

   Knowing that the alarm is going off, if a burglary is happening, then it is less likely that an earthquake is happening.