Probabilities and Independence

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Lecture 9

Based on work by K. Leyton-Brown, K. Larson, and P. van Beek

Outline

Learning Goals

By the end of the lecture, you should be able to

- Calculate prior, posterior, and joint probabilities using the sum rule, the product rule, the chain rule and Bayes' rule.

Why handle uncertainty?

Why does an agent need to handle uncertainty?

- An agent may not observe everything in the world.
- An action may not have its intended consequences.

An agent needs to

- Reason about its uncertainty.
- Make a decision based on their uncertainty.

Probability

- Probability is the formal measure of uncertainty.
- There are two camps: Frequentists and Bayesians.
- Frequentists' view of probability:
    - Frequentists view probability as something *objective*.
    - Compute probabilities by counting the frequencies of events.
- Bayesians' view of probability:
    - Bayesians view probability as something *subjective*.
    - Probabilities are degrees of belief.
    - We start with prior beliefs and update beliefs based on new evidence.

Random variable

A random variable

- Has a domain of possible values
- Has an associated probability distribution, which is a function from the domain of the random variable to $[0, 1]$.

Example:

- random variable: The alarm is going.
- domain: {true, false}
- P(The alarm is going = true) = 0.1
  P(The alarm is going = false) = 0.9

Shorthand notation

Let A and B be Boolean random variables.

- $P(A)$ denotes $P(A = true)$.
- $P(\neg A)$ denotes $P(A = false)$.

Differentiate $P(A, B)$ and $P(A \wedge B)$.

- $P(A, B)$ denotes the joint distribution of A and B.
- $P(A \wedge B)$ denotes $P(A = true, B = true)$ - one value in the joint distribution.

Axioms of Probability

Let A and B be Boolean random variables.

- Every probability is between 0 and 1.

$$0 \leq P(A) \leq 1$$

- Necessarily true propositions have prob 1. Necessarily false propositions have probability 0.

$$P(true) = 1, P(false) = 0$$

- The inclusion-exclusion principle:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

These axioms limit the functions that can be considered as probability functions.

Joint Probability Distribution

- A probabilistic model contains a set of random variables.
- An atomic event assigns a value to every random variable in the model.
- A joint probability distribution assigns a probability to every atomic event.

Prior and Posterior Probabilities

$P(X)$:

- prior or unconditional probability
- Likelihood of X in the absence of any other information
- Based on the background information

$P(X|Y)$

- posterior or conditional probability
- Likelihood of X given Y.
- Based on Y as evidence

The Holmes Scenario

Mr. Holmes lives in a high crime area and therefore has installed a burglar alarm. He relies on his neighbors to phone him when they hear the alarm sound. Mr. Holmes has two neighbors, Dr. Watson and Mrs. Gibbon.

Unfortunately, his neighbors are not entirely reliable. Dr. Watson is known to be a tasteless practical joker and Mrs. Gibbon, while more reliable in general, has occasional drinking problems.

Mr. Holmes also knows from reading the instruction manual of his alarm system that the device is sensitive to earthquakes and can be triggered by one accidentally. He realizes that if an earthquake has occurred, it would surely be on the radio news.

The Holmes Scenario - A Trimmed Version

Mr. Holmes has installed a burglar alarm. He relies on his neighbors to phone him when they hear the alarm sound. Mr. Holmes has two neighbors, Dr. Watson and Mrs. Gibbon.

Unfortunately, his neighbors are not entirely reliable. Dr. Watson is known to be a tasteless practical joker and Mrs. Gibbon, while more reliable in general, has occasional drinking problems.

Modeling the Holmes Scenario

What are the random variables?

How many probabilities are there in the joint probability distribution?

The Joint Distribution

A			$\neg A$		
	G	$\neg G$		G	$\neg G$
W	0.032	0.048	W	0.036	0.324
$\neg W$	0.008	0.012	$\neg W$	0.054	0.486

CQ: Calculating a joint probability

CQ: What is probability that
the alarm is NOT going and Dr. Watson is calling?

(A) $0 \leq p \leq 0.2$

(B) $0.2 < p \leq 0.4$

(C) $0.4 < p \leq 0.6$

(D) $0.6 < p \leq 0.8$

(E) $0.8 < p \leq 1$

CQ: Calculating a joint probability

CQ: What is probability that
the alarm is going and Mrs. Gibbon is NOT calling?

(A) $0 \leq p \leq 0.2$

(B) $0.2 < p \leq 0.4$

(C) $0.4 < p \leq 0.6$

(D) $0.6 < p \leq 0.8$

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CQ: Calculating a probability

CQ: What is probability that **the alarm is NOT going**?

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(B) $0.2 < p \leq 0.4$

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The Sum Rule

Given a joint distribution, we can compute distributions over a subset of the variables.

Example 1:

$$P(X = a) = \sum_{b \in dom(Y)} P(X = a \wedge Y = b)$$

Example 2:

$$P(X = a \wedge Y = b) \quad (1)$$

$$= \sum_{c \in dom(Z)} \sum_{d \in dom(M)} P(X = a \wedge Y = b \wedge Z = c \wedge M = d) \quad (2)$$

CQ: Calculating a conditional probability

CQ: What is probability that
Dr. Watson is calling given that the alarm is NOT going?

(A) $0 \leq p \leq 0.2$

(B) $0.2 < p \leq 0.4$

(C) $0.4 < p \leq 0.6$

(D) $0.6 < p \leq 0.8$

(E) $0.8 < p \leq 1$

CQ: Calculating a conditional probability

CQ: What is probability that
Mrs. Gibbon is NOT calling given that the alarm is going?

(A) $0 \leq p \leq 0.2$

(B) $0.2 < p \leq 0.4$

(C) $0.4 < p \leq 0.6$

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The Product Rule

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} \text{ whenever } P(Y) > 0$$

The Prior and Conditional Probabilities

The prior probabilities:

$P(A) = 0.1$ $\qquad P(W) = 0.4$ $\qquad P(G) = 0.1$

The conditional probabilities

$P(W|A) = 0.9$ $\qquad P(G|A) = 0.3$
$P(W|\neg A) = 0.4$ $\qquad P(G|\neg A) = 0.1$

$P(W|A \wedge G) = 0.9$ $\qquad P(G|A \wedge W) = 0.3$
$P(W|A \wedge \neg G) = 0.9$ $\qquad P(G|A \wedge \neg W) = 0.3$
$P(W|\neg A \wedge G) = 0.4$ $\qquad P(G|\neg A \wedge W) = 0.1$
$P(W|\neg A \wedge \neg G) = 0.4$ $\qquad P(G|\neg A \wedge \neg W) = 0.1$

The Chain Rule

The chain rule for two variables (a.k.a. the product rule):

$$P(A, B) = P(A|B) * P(B)$$

The chain rule for three variables:

$$P(A, B, C) = P(A|B, C) * P(B|C) * P(C)$$

The chain rule can be generalized to any number of variables.

$$\begin{aligned} &P(X_n, X_{n-1}, ..., X_2, X_1) \\ &= \prod_{i=1}^{n} P(X_i|X_{i-1}, \ldots, X_1) \\ &= P(X_n|X_{n-1}, ..., X_2, X_1) * ... * P(X_2|X_1) * P(X_1) \end{aligned}$$

CQ: Calculating the joint probability

CQ: What is probability that **the alarm is going, Dr. Watson is calling and Mrs. Gibbon is NOT calling**?

(A) $0 \leq p \leq 0.2$

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CQ: Calculating the joint probability

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Bayes' Rule

Definition (Bayes' rule)

$$P(X|Y) = \frac{P(Y|X) * P(X)}{P(Y)}.$$

Why is Bayes' rule useful?

Often you have causal knowledge:

- $P(symptom \mid disease)$
- $P(alarm \mid fire)$

...and you want to do evidential reasoning:

- $P(disease \mid symptom)$
- $P(fire \mid alarm)$.

CQ Applying the Bayes' Rule

CQ: What is the probability that
the alarm is NOT going given that Dr. Watson is calling?

(A) $0 \leq p \leq 0.2$

(B) $0.2 < p \leq 0.4$

(C) $0.4 < p \leq 0.6$

(D) $0.6 < p \leq 0.8$

(E) $0.8 < p \leq 1$

CQ Applying the Bayes' Rule

CQ: What is the probability that
the alarm is going given that Mrs. Gibbon is NOT calling?

(A) $0 \leq p \leq 0.2$

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Revisiting the Learning Goals

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