Semantic Entailment and Natural Deduction

Alice Gao

Lecture 6, September 26, 2017

Learning goals

Semantic entailment

- Define semantic entailment.
- Explain subtleties of semantic entailment.
- Determine whether a semantic entailment holds by using truth tables, valuation trees, and/or logical identities.
- Prove semantic entailment using truth tables and/or valuation trees.

Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

A review of the conditional

Consider the formulas $p_1 \wedge p_2 \wedge p_3$ and c.

The following two statements are equivalent:

- for any truth valuation t, if $(p_1 \wedge p_2 \wedge p_3)$ is true, then c is true.
- $(p_1 \wedge p_2 \wedge p_3) \rightarrow c$ is a tautology.

Subtleties about the conditional

Consider the formulas $p_1 \wedge p_2 \wedge p_3$ and c. How many of the following statements are true?

a. If p_1 is false, then $(p_1 \wedge p_2 \wedge p_3) \rightarrow c$ is true.
b. If $p_1 = x$ and $p_2 = (\neg x)$, then $(p_1 \wedge p_2 \wedge p_3) \rightarrow c$ is false.
c. If c is a tautology, then $(p_1 \wedge p_2 \wedge p_3) \rightarrow c$ is true.
d. Two of (a), (b), and (c) are true.
e. All of (a), (b), and (c) are true.

Proving arguments valid

Recall that logic is the science of reasoning.

One important goal of logic is to infer that a conclusion is true based on a set of premises.

A logical argument:

Premise 1

Premise 2

...

Premise n

Conclusion

A common problem is to prove that an argument is valid, that is the set of premises semantically entails the conclusion.

Formalizing argument validity: Semantic Entailment

Let $\Sigma = \{p_1, p_2, ..., p_n\}$ be a set of premises and let α be the conclusion that we want to derive.

Σ *semantically entails* α, denoted $\Sigma \vDash \alpha$, if and only if

- Whenever all the premises in Σ are true, then the conclusion α is true.
- For any truth valuation t, if every premise in Σ is true under t, then the conclusion α is true under t.
- For any truth valuation t, if t satisfies Σ (denoted $\Sigma^t = \mathtt{T}$), then t satisfies α ($\alpha^t = \mathtt{T}$).
- $(p_1 \wedge p_2 \wedge ... \wedge p_n) \rightarrow \alpha$ is a tautology.

If Σ semantically entails α, then we say that the argument (with the premises in Σ and the conclusion α) is valid.

What does $\Sigma^t = \mathtt{T}$ (t satisfies Σ) mean? See the next slide.

What does $\Sigma^t = \texttt{T}$ mean?

$\Sigma^t = \texttt{T}$ (t satisfies Σ) means ...

- Every formula in Σ is true under the valuation t.
- If a formula β is in Σ, then β is true under t.

If Σ is the empty set $\emptyset$, then any valuation satisfies Σ. Why?

The definition of "t satisfies Σ" says

- If a formula β is in Σ, then β is true under t.

There is no formula in $\emptyset$, so the premise of the above statement is false, which means the statement is vacuously true. Thus, any valuation satisfies the empty set $\emptyset$.

Subtleties about entailment

Consider a set of formulas Σ and the formula α. How many of the following statements are true?

a. If p_1 in Σ is false, then $\Sigma \vDash \alpha$ is false.
b. If $\Sigma = \{x, (\neg x)\}$, then $\Sigma \vDash \alpha$ is true.
c. If $\emptyset \vDash \alpha$ is true, then α is a tautology ($\emptyset$ is the empty set).
d. Two of (a), (b), and (c) are true.
e. All of (a), (b), and (c) are true.

Proving or disproving entailment

Proving that Σ entails α, denoted $\Sigma \models \alpha$:

- Using a truth table: Consider all rows of the truth table in which all of the formulas in Σ are true. Verify that α is true in all of these rows.
- Direct proof: For every truth valuation under which all of the premises are true, show that the conclusion is also true under this valuation.
- Proof by contradiction: Assume that the entailment does not hold, which means that there is a truth valuation under which all of the premises are true and the conclusion is false. Derive a contradiction.

Proving that Σ does not entail α, denoted $\Sigma \not\models \alpha$:

- Find one truth valuation t under which all of the premises in Σ are true and the conclusion α is false.

Proving entailment using a truth table

Let $\Sigma = \{(\neg(p \wedge q)), (p \to q)\}$, $x = (\neg p)$, and $y = (p \leftrightarrow q)$. Based on the truth table, which of the following statements is true?

a. $\Sigma \models x$ and $\Sigma \models y$.

b. $\Sigma \models x$ and $\Sigma \not\models y$.

c. $\Sigma \not\models x$ and $\Sigma \models y$.

d. $\Sigma \not\models x$ and $\Sigma \not\models y$.

p	q	$(\neg(p \wedge q))$	$(p \to q)$	$x = (\neg p)$	$y = (p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	1	0
1	0	1	0	0	0
1	1	0	1	0	1

Proving entailment

What is $\{(\neg(p \wedge q)), (p \wedge q)\} \vDash (p \leftrightarrow q)$?

a. True

b. False

p	q	$(\neg(p \wedge q))$	$(p \wedge q)$	$(p \leftrightarrow q)$
0	0	1	0	1
0	1	1	0	0
1	0	1	0	0
1	1	0	1	1

Equivalence and Entailment

Equivalence can be expressed using the notion of entailment.

Lemma. $\alpha \equiv \beta$ if and only if both $\{\alpha\} \models \beta$ and $\{\beta\} \models \alpha$.

Proofs in Propositional Logic: Natural Deduction

Solution to the previous puzzle

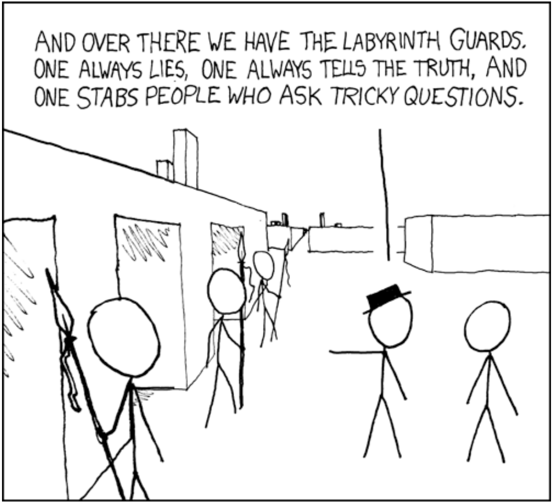
A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

You meet three inhabitants: Alice, Rex and Bob.

1. Alice says, "Rex is a knave." *This means Alice and Rex are different.*
2. Rex says, "it's false that Bob is a knave (or Bob is a knight)." *This means Rex and Bob are the same.*
3. Bob claims, "I am a knight or Alice is a knight." *Bob is a knight, or Bob and Alice are both knaves.*

Based on 1 and 2, Alice and Bob are different, so they cannot both be knaves (2nd option in 3). Thus, the only possibility left is Alice is a knave, and Rex and Bob are knights.

Labyrinth Puzzle



Learning goals

Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

The Natural Deduction Proof System

We will consider a proof system called Natural Deduction.

- It closely follows how people (mathematicians, at least) normally make formal arguments.
- It extends easily to more-powerful forms of logic.

Why would you want to study natural deduction proofs?

- It is impressive to be able to write proofs with nested boxes and mysterious symbols as justifications.
- Be able to prove or disprove that Superman exists (on Tuesday).
- Be able to prove or disprove that the onnagata are correct to insist that males should play female characters in Japanese kabuki theatres.
- To realize that writing proofs and problem solving in general is both a creative and a scientific endeavour.
- To develop problem solving strategies that can be used in many other situations.

A proof is syntactic

First, we think about proofs in a purely syntactic way.

A proof

- starts with a set of premises,
- transforms the premises based on a set of inference rules (by pattern matching),
- and reaches a conclusion.

We write

$$\Sigma \vdash_{ND} \varphi \quad \text{or simply} \quad \Sigma \vdash \varphi$$

if we can find such a proof that starts with a set of premises Σ and ends with the conclusion φ.

Goal is to show semantic entailment

Next, we think about connecting proofs to semantic entailment.

We will answer these questions:

- (Soundness) Does every proof establish a semantic entailment?
  If I can find a proof from Σ to φ, can I conclude that Σ semantically entails φ?
  Does $\Sigma \vdash \varphi$ imply $\Sigma \vDash \varphi$?
- (Completeness) For every semantic entailment, can I find a proof for it?
  If I know that Σ semantically entails φ, can I find a proof from Σ to φ?
  Does $\Sigma \vDash \varphi$ imply $\Sigma \vdash \varphi$?

Reflexivity / Premise

If you want to write down a previous formula in the proof again, you can do it by *reflexivity*.

Name	**$\vdash$-notation**	**inference notation**
Reflexivity, or Premise	$\Sigma, \alpha \vdash \alpha$	$\dfrac{\alpha}{\alpha}$

The notation on the right: Given the formulas above the line, we can infer the formula below the line.

The version in the center reminds us of the role of assumptions in Natural Deduction. Other rules will make more use of it.

An example using reflexivity

Here is a proof of $\{p, q\} \vdash p$.

1.	p	Premise
2.	q	Premise
3.	p	Reflexivity: 1

Alternatively, we could simply write

1.	p	Premise

and be done.

For each symbol, the rules come in pairs.

- An "introduction rule" adds the symbol to the formula.
- An "elimination rule" removes the symbol from the formula.

Rules for Conjunction

Name	**$\vdash$-notation**	**inference notation**
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$, then $\Sigma \vdash (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	**$\vdash$-notation**	**inference notation**
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash (\alpha \wedge \beta)$, then $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash (q \wedge p)$.

1.	$(p \wedge q)$	Premise
2.	q	$\wedge$e: 1
3.	p	$\wedge$e: 1
4.	$(q \wedge p)$	$\wedge$i: 2, 3

Example: Conjunction Rules (2)

Example. Show that $\{(p \land q), r\} \vdash (q \land r)$.

1.	$(p \land q)$	Premise
2.	r	Premise
3.	q	$\land$e: 1
4.	$(q \land r)$	$\land$i: 3, 2

Rules for Implication: $\rightarrow$e

Name	**$\vdash$-notation**	**inference notation**
$\rightarrow$-elimination ($\rightarrow$e) (modus ponens)	If $\Sigma \vdash (\alpha \rightarrow \beta)$ and $\Sigma \vdash \alpha$, then $\Sigma \vdash \beta$	$\dfrac{(\alpha \rightarrow \beta) \quad \alpha}{\beta}$

In words:

If you assume α is true and α implies β, then you may conclude β.

Rules for Implication: $\rightarrow$i

Name	**$\vdash$-notation**	**inference notation**
$\rightarrow$-introduction ($\rightarrow$i)	If $\Sigma, \alpha \vdash \beta$, then $\Sigma \vdash (\alpha \rightarrow \beta)$	$\dfrac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}}}{(\alpha \rightarrow \beta)}$

The "box" denotes a sub-proof. In the sub-proof, we starts by assuming that α is true (a premise of the sub-proof), and we conclude that β is true.

Nothing inside the sub-proof may come out.

Outside of the sub-proof, we could only use the sub-proof as a whole.

Example: Rule $\rightarrow$i and sub-proofs

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$.

To start, we write down the premises at the beginning, and the conclusion at the end.

1.	$(p \rightarrow q)$	Premise
2.	$(q \rightarrow r)$	Premise
3.	p	Assumption
4.	q	$\rightarrow$e: 1, 3
5.	r	$\rightarrow$e: 2, 4
6.	$(p \rightarrow r)$	???

What next?

The goal "$(p \rightarrow r)$" contains $\rightarrow$. Let's try rule $\rightarrow$i....

Inside the sub-proof, we can use rule $\rightarrow$e.

Done!

Rules of Disjunction: $\vee$i and $\vee$e

Name	**$\vdash$-notation**	**inference notation**
$\vee$-introduction ($\vee$i)	If $\Sigma \vdash \alpha$, then $\Sigma \vdash \alpha \vee \beta$ and $\Sigma \vdash \beta \vee \alpha$	$\dfrac{\alpha}{\alpha \vee \beta}$ $\quad$ $\dfrac{\alpha}{\beta \vee \alpha}$
$\vee$-elimination ($\vee$e)	If $\Sigma, \alpha_1 \vdash \beta$ and $\Sigma, \alpha_2 \vdash \beta$, then $\Sigma, \alpha_1 \vee \alpha_2 \vdash \beta$	$\dfrac{\alpha_1 \vee \alpha_2 \quad \boxed{\begin{array}{c}\alpha_1\\ \vdots\\ \beta\end{array}} \quad \boxed{\begin{array}{c}\alpha_2\\ \vdots\\ \beta\end{array}}}{\beta}$

$\vee$e is also known as “proof by cases”.

Example: Or-Introduction and -Elimination

Example: Show that $\{p \lor q\} \vdash (p \to q) \lor (q \to p)$.

1.	$p \lor q$	Premise
2.	p	Assumption
3.	q	Assumption
4.	p	Reflexivity: 2
5.	$q \to p$	$\to$i: 3–4
6.	$(p \to q) \lor (q \to p)$	$\lor$i: 5
7.	q	Assumption
8.	p	Assumption
9.	q	Reflexivity: 7
10.	$p \to q$	$\to$i: 8–9
11.	$(p \to q) \lor (q \to p)$	$\lor$i: 10
12.	$(p \to q) \lor (q \to p)$	$\lor$e: 1, 2–6, 7–11

Negation

We shall treat negation by considering contradictions.

We shall use the notation $\perp$ to represent any contradiction.
It may appear in proofs as if it were a formula.

The elimination rule for negation:

Name	**$\vdash$-notation**	**inference notation**
$\perp$-introduction, or $\neg$-elimination ($\neg$e)	$\Sigma,\ \alpha,\ (\neg\alpha) \vdash \perp$	$\dfrac{\alpha \quad (\neg\alpha)}{\perp}$

If we have both α and $(\neg\alpha)$, then we have a contradiction.

Negation Introduction (¬i)

If an assumption α leads to a contradiction, then derive $(\neg\alpha)$.

Name	**$\vdash$-notation**	**inference notation**
¬-introduction (¬i)	If $\Sigma, \alpha \vdash \bot$, then $\Sigma \vdash (\neg\alpha)$	$\dfrac{\boxed{\begin{array}{c}\alpha \\ \vdots \\ \bot\end{array}}}{(\neg\alpha)}$

Example: Negation

Example. Show that $\{\alpha \to (\neg\alpha)\} \vdash (\neg\alpha)$.

1.	$\alpha \to (\neg\alpha)$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\to$e: 1, 2
4.	$\perp$	$\neg$e: 2, 3
5.	$(\neg\alpha)$	$\neg$i: 2–4

The Last Two Basic Rules

Double-Negation Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\neg\neg$-elimination ($\neg\neg$e)	If $\Sigma \vdash (\neg(\neg\alpha))$, then $\Sigma \vdash \alpha$	$\dfrac{(\neg(\neg\alpha))}{\alpha}$

Contradiction Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\bot$-elimination ($\bot$e)	If $\Sigma \vdash \bot$, then $\Sigma \vdash \alpha$	$\dfrac{\bot}{\alpha}$

A Redundant Rule

The rule of $\bot$-elimination is not actually needed.

Suppose a proof has

27.	$\bot$	$\langle$*some rule*$\rangle$
28.	α	$\bot$e: 27.

We can replace these by

27.	$\bot$	$\langle$*some rule*$\rangle$
28.	$(\neg\alpha)$	Assumption
29.	$\bot$	Reflexivity: 27
30.	$(\neg(\neg\alpha))$	$\neg$i: 28–29
31.	α	$\neg\neg$e: 30.

Thus any proof that uses $\bot$e can be modified into a proof that does not.

Example: "*Modus tollens*"

The principle of *modus tollens*: $\{p \to q, (\neg q)\} \vdash (\neg p)$.

1.	$p \to q$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.	q	$\to$e: 3, 1
5.	$\bot$	$\neg$e: 2, 4
6.	$(\neg p)$	??

Modus tollens is sometimes taken as a "derived rule":

$$\frac{\alpha \to \beta \quad (\neg \beta)}{(\neg \alpha)} \text{ MT}$$

Derived Rules

Whenever we have a proof of the form $\Gamma \vdash \alpha$, we can consider it as a derived rule:

$$\frac{\Gamma}{\alpha}$$

If we use this in a proof, it can be replaced by the original proof of $\Gamma \vdash \alpha$. The result is a proof using only the basic rules.

Using derived rules does not expand the things that can be proved. But they can make it easier to find a proof.

Strategies for natural deduction proofs

1. Work forward from the premises. Can you apply an elimination rule?
2. Work backwards from the conclusion. What introduction rule do you need to use at the end?
3. Stare at the formula. Notice its structure. Use it to guide your proof.
4. If a direct proof doesn't work, try a proof by contradiction.

Further Examples of Natural Deduction

Example. Show that $\{p \to q\} \vdash (r \vee p) \to (r \vee q)$.

In the sub-proof, try $\vee$-elimination on the assumption (step 2).

1.	$p \to q$	Premise
2.	$r \vee p$	Assumption
3.	r	Assumption
4.	$r \vee q$	$\vee$i: 3
5.	p	Assumption
6.	q	$\to$e: 5, 1
7.	$r \vee q$	$\vee$i: 6
8.	$r \vee q$	$\vee$e: 2, 3–4, 5–7
9.	$(r \vee p) \to (r \vee q)$	$\to$i: 2–8

Life's Not Always So Easy...

Example. Show that $\vdash \big((p \to q) \to p\big) \to p$.

1.	$(p \to q) \to p$	Assumption
2.		*No elimination applies.*
3.		
4.	?????	
5.	p	*No connective.*
6.	$\big((p \to q) \to p\big) \to p$	*Try* $\to$*i...*

Time to try something ingenious....

Some Common Derived Rules

Proof by contradiction (*reductio ad absurdum*):

$$\text{if } \Sigma, (\neg\alpha) \vdash \bot, \text{ then } \Sigma \vdash \alpha.$$

The "Law of Excluded Middle" (*tertiam non datur*): $\vdash \alpha \vee (\neg\alpha)$.

Double-Negation Introduction: if $\Sigma \vdash \alpha$ then $\Sigma \vdash (\neg(\neg\alpha))$.

You can try to prove these yourself, as exercises.
(Hint: in the first two, the last step uses rule $\neg\neg$e: $(\neg(\neg\alpha)) \vdash \alpha$.)

Or see pages 24–26 of Huth and Ryan.

Soundness and Completeness of Natural Deduction

for Propositional Logic

Soundness and Completeness of Natural Deduction

We want to prove that Natural Deduction is both sound and complete.

Soundness of Natural Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

If $\Sigma \vdash_{ND} \alpha$, then $\Sigma \vDash \alpha$.

Completeness of Natural Deduction means that all logical consequences in propositional logic are provable in Natural Deduction. That is,

If $\Sigma \vDash \alpha$, then $\Sigma \vdash_{ND} \alpha$.

Proof of Soundness

To prove soundness, we use induction on the *length of the proof*:

> For all deductions $\Sigma \vdash \alpha$ which have a proof of length n or less, it is the case that $\Sigma \models \alpha$.

That property, however, is not quite good enough to carry out the induction. We actually use the following property of a natural number n.

> Suppose that a formula α appears at line n of a partial deduction, which may have one or more open sub-proofs. Let Σ be the set of premises used and Γ be the set of assumptions of open sub-proofs. Then $\Sigma \cup \Gamma \models \alpha$.

Basis of the Induction

Base case. The shortest deductions have length 1, and thus are either

1.	α	Premise.

or

1.	α	Assumption.
2.		

We have either $\alpha \in \Sigma$ (in the first case), or $\alpha \in \Gamma$ (in the second case).

Thus $\Sigma \cup \Gamma \vDash \alpha$, as required.

Proof of Soundness: Inductive Step

Inductive step. Hypothesis: the property holds for each $n < k$; that is,

> If some formula α appears at line k or earlier of some partial deduction, with premises Σ and un-closed assumptions Γ, then $\Sigma \cup \Gamma \models \alpha$.

To prove: if α' appears at line $k+1$, then $\Sigma \cup \Gamma' \models \alpha'$
(where $\Gamma' = \Gamma \cup \alpha'$ when α' is an assumption, and $\Gamma' = \Gamma$ otherwise).

The case that α' is an assumption is trivial.

Otherwise, formula α' must have a justification by some rule. We shall consider each possible rule.

Inductive Step, Case I

Case I: α' was justified by $\wedge$i.

We must have $\alpha' = \alpha_1 \wedge \alpha_2$, where each of α_1 and α_2 appear earlier in the proof, at steps m_1 and m_2, respectively. Also, any sub-proof open at step m_1 or m_2 is still open at step $k + 1$.

Thus the induction hypothesis applies to both; that is, $\Sigma \cup \Gamma \vDash \alpha_1$ and $\Sigma \cup \Gamma \vDash \alpha_2$.

By the definition of $\vDash$, this yields $\Sigma \cup \Gamma \vDash \alpha'$, as required.

Inductive Step, Case II

Case II: α' was justified by $\rightarrow$i.

Rule $\rightarrow$i requires that $\alpha' = \alpha_1 \rightarrow \alpha_2$ and there is a closed sub-proof with assumption α_1 and conclusion α_2, ending by step k. Also, any sub-proof open before the assumption of α_1 is still open at step $k+1$.

The induction hypothesis thus implies $\Sigma \cup (\Gamma \cup \{\alpha_1\}) \vDash \alpha_2$.

Hence $\Sigma \cup \Gamma \vDash \alpha_1 \rightarrow \alpha_2$, as required.

Inductive Step, Cases III ff.

Case III: α' was justified by $\neg$e.

This requires that α' be the pseudo-formula $\bot$, and that the proof contain formulas α and $(\neg\alpha)$ for some α, each using at most k steps.

By the induction hypothesis, both $\Sigma \vDash \alpha$ and $\Sigma \vDash (\neg\alpha)$.

Thus Σ is contradictory, and $\Sigma \vDash \alpha'$ for any α'.

Cases IV–XIII:

The other cases follow by similar reasoning.

This completes the inductive step, and the proof of soundness.

Completeness of Natural Deduction

We now turn to completeness.

Recall that *completeness* means the following.

Let Σ be a set of formulas and φ be a formula.

$$\text{If } \Sigma \vDash \varphi, \text{ then } \Sigma \vdash \varphi \ .$$

That is, every consequence has a proof.

How can we prove this?

Proof of Completeness: Getting started

We shall assume that the set Σ of hypotheses is finite.
The theorem is also true for infinite sets of hypotheses, but that requires a completely different proof.

Suppose that $\Sigma \models \varphi$, where $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$.
Thus the formula $(\sigma_1 \wedge \sigma_2 \wedge \dots \wedge \sigma_m) \rightarrow \varphi$ is a tautology.

Lemma. Every tautology is provable in Natural Deduction.

Once we prove the Lemma, the result follows. Given a proof of $(\sigma_1 \wedge \sigma_2 \wedge \dots \wedge \sigma_m) \rightarrow \varphi$, one can use $\wedge$i and $\rightarrow$e to complete a proof of $\Sigma \vdash \varphi$.

Tautologies Have Proofs

For a tautology, every line of its truth table ends with T.
We can mimic the construction of a truth table using inferences in Natural Deduction.

Claim. Let φ have k variables $p_1, \dots, p_k$. Let v be a valuation, and define $\ell_1, \ell_2, \dots, \ell_k$ as

$$\ell_i = \begin{cases} p_i & \text{if } v(p_i) = \mathtt{T} \\ \neg p_i & \text{if } v(p_i) = \mathtt{F}. \end{cases}$$

If $\varphi^v = \mathtt{T}$, then $\{\ell_1, \dots \ell_k\} \vdash \varphi$, and
if $\varphi^v = \mathtt{F}$, then $\{\ell_1, \dots \ell_k\} \vdash (\neg\varphi)$.

To prove the claim, use structural induction on formulas (which is induction on the column number of the truth table).

Once the claim is proven, we can prove a tautology as follows....

Outline of the Proof of a Tautology

1.	$p_1 \vee (\neg p_1)$	L.E.M.
2.	$p_2 \vee (\neg p_2)$	L.E.M.
$\vdots$	$\vdots$	
k.	$p_k \vee (\neg p_k)$	L.E.M.
$k+1$.	p_1	assumption
	p_2	assumption
	$\vdots$	
	φ	
	$(\neg p_2)$	assumption
	$\vdots$	
	φ	
m.	φ	$\vee$e: 2, ...

$m+1$.	$(\neg p_1)$	assumption
	$\vdots$	
	$\vdots$	
	φ	$\vee$e: $m+1$, ...
n.	φ	$\vee$e: 1, $m-(k+1)$, $n-(m+1)$

Once each variable is assumed true or false, the previous claim provides a proof.

Proving the Claim

Hypothesis: the following hold for formulas α and β:

> If $\{\ell_1, \dots, \ell_k\} \vDash \alpha$, then $\{\ell_1, \dots, \ell_k\} \vdash \alpha$;
> If $\{\ell_1, \dots, \ell_k\} \nvDash \alpha$, then $\{\ell_1, \dots, \ell_k\} \vdash (\neg\alpha)$;
> If $\{\ell_1, \dots, \ell_k\} \vDash \beta$, then $\{\ell_1, \dots, \ell_k\} \vdash \beta$; and
> If $\{\ell_1, \dots, \ell_k\} \nvDash \beta$, then $\{\ell_1, \dots, \ell_k\} \vdash (\neg\beta)$.

If $\{\ell_1, \dots, \ell_k\} \vDash (\alpha \land \beta)$, put the two proofs of α and β together, and then infer $(\alpha \land \beta)$, by $\land$i.

If $\{\ell_1, \dots, \ell_k\} \nvDash (\alpha \to \beta)$ (i.e., $\{\ell_1, \dots, \ell_k\} \vDash \alpha$ and $\{\ell_1, \dots, \ell_k\} \nvDash \beta$),

- Prove α and $(\neg\beta)$.
- Assume $(\alpha \to \beta)$; from it, conclude β ($\to$e) and then $\perp$ ($\neg$e).
- From the sub-proof, conclude $\big(\neg(\alpha \to \beta)\big)$, by $\neg$i.

The other cases are similar.