

# *Propositional Logic: Semantic Entailment*

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Lecture 6

# Learning goals

## Semantic entailment

- Determine if a set of formulas is satisfiable.
- Define semantic entailment.
- Explain subtleties of semantic entailment.
- Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.

# Proving arguments valid

Logic is the science of reasoning.

One goal of logic is to perform deductions — to infer that a conclusion is true based on a set of premises that we know to be true.

The process of logical deduction is formalized by the notion of semantic entailment.

Can we show that the conclusion semantically follows from the set of premises?

# Semantic Entailment

Let  $\Sigma$  be a set of premises. Let  $\varphi$  be the conclusion.

*Definition.* A truth valuation  $t$  satisfies  $\Sigma$  (denoted  $\Sigma^t = \mathbf{T}$ ) if and only if for any formula  $\alpha$ , if  $\alpha \in \Sigma$ , then  $\alpha^t = \mathbf{T}$ .

*Definition.*  $\Sigma$  semantically entails  $\varphi$  (denoted  $\Sigma \models \varphi$ ) if and only if for any truth valuation  $t$ , if  $\Sigma^t = \mathbf{T}$ , then  $\varphi^t = \mathbf{T}$ .

# Prove an entailment

Consider the entailment  $\Sigma \models \varphi$ . To prove that the entailment holds, we need to consider

- A) Every truth valuation  $t$  under which  $\Sigma^t = T$ .
- B) Every truth valuation  $t$  under which  $\Sigma^t = F$ .
- C) One truth valuation  $t$  under which  $\Sigma^t = T$ .
- D) One truth valuation  $t$  under which  $\Sigma^t = F$ .

# A semantic entailment does not hold

Let  $\Sigma$  be a set of premises. Let  $\varphi$  be the conclusion.

*Definition.*  $\Sigma$  semantically entails  $\varphi$  (denoted  $\Sigma \models \varphi$ ) if and only if

for any truth valuation  $t$ , if  $\Sigma^t = \mathbf{T}$ , then  $\varphi^t = \mathbf{T}$ .

*Definition.*  $\Sigma$  does not entail  $\varphi$  (denoted  $\Sigma \not\models \varphi$ ) if and only if

there exists a truth valuation  $t$  such that  $\Sigma^t = \mathbf{T}$  and  $\varphi^t = \mathbf{F}$ .

# Disprove an entailment

Consider the entailment  $\Sigma \models \varphi$ . To prove that the entailment does NOT hold, we need to find

- A) Every truth valuation  $t$  under which  $\Sigma^t = \text{true}$  and  $\varphi = \text{T}$ .
- B) Every truth valuation  $t$  under which  $\Sigma^t = \text{true}$  and  $\varphi = \text{F}$ .
- C) One truth valuation  $t$  under which  $\Sigma^t = \text{true}$  and  $\varphi = \text{T}$ .
- D) One truth valuation  $t$  under which  $\Sigma^t = \text{true}$  and  $\varphi = \text{F}$ .

# Proving/disproving an entailment using a truth table

Let  $\Sigma = \{(\neg(p \wedge q)), (p \rightarrow q)\}$ ,  $x = (\neg p)$ , and  $y = (p \leftrightarrow q)$ . Based on the truth table, which of the following statements is true?

- A)  $\Sigma \models x$  and  $\Sigma \models y$ .
- B)  $\Sigma \models x$  and  $\Sigma \not\models y$ .
- C)  $\Sigma \not\models x$  and  $\Sigma \models y$ .
- D)  $\Sigma \not\models x$  and  $\Sigma \not\models y$ .

$p$	$q$	$(\neg(p \wedge q))$	$(p \rightarrow q)$	$x = (\neg p)$	$y = (p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	1	0
1	0	1	0	0	0
1	1	0	1	0	1



# Proving/disproving an entailment using the definition

*Exercise.* Show that  $\{(\neg(p \wedge q)), (p \rightarrow q)\} \models (\neg p)$ .

*Exercise.* Show that  $\{(\neg(p \wedge q)), (p \rightarrow q)\} \not\models (p \leftrightarrow q)$ .

# Subtleties of entailment

Consider the entailment  $\Sigma \models \varphi$ .

Does the entailment hold under each of the following conditions?

1.  $\Sigma$  is the empty set.
2.  $\Sigma$  is not satisfiable.
3.  $\varphi$  is a tautology.
4.  $\varphi$  is a contradiction.

# *Propositional Logic:*

## *Natural Deduction*

# Learning goals

## Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

# The Natural Deduction Proof System

We will consider a proof system called Natural Deduction.

- It closely follows how people (mathematicians, at least) normally make formal arguments.
- It extends easily to more-powerful forms of logic.

# Why would you want to study natural deduction proofs?

- It is impressive to be able to write proofs with nested boxes and mysterious symbols as justifications.
- Be able to prove or disprove that Superman exists (on Tuesday).
- Be able to prove or disprove that the onnagata are correct to insist that males should play female characters in Japanese kabuki theatres.
- To realize that writing proofs and problem solving in general is both a creative and a scientific endeavour.
- To develop problem solving strategies that can be used in many other situations.

# A proof is syntactic

First, we think about proofs in a purely syntactic way.

A proof

- starts with a set of premises,
- transforms the premises based on a set of inference rules (by pattern matching),
- and reaches a conclusion.

We write

$$\Sigma \vdash_{ND} \varphi \quad \text{or simply} \quad \Sigma \vdash \varphi$$

if we can find such a proof that starts with a set of premises  $\Sigma$  and ends with the conclusion  $\varphi$ .

# Goal is to show semantic entailment

Next, we think about connecting proofs to semantic entailment.

We will answer these questions:

- (Soundness) Does every proof establish a semantic entailment?  
If I can find a proof from  $\Sigma$  to  $\varphi$ , can I conclude that  $\Sigma$  semantically entails  $\varphi$ ?  
Does  $\Sigma \vdash \varphi$  imply  $\Sigma \models \varphi$ ?
- (Completeness) For every semantic entailment, can I find a proof for it?  
If I know that  $\Sigma$  semantically entails  $\varphi$ , can I find a proof from  $\Sigma$  to  $\varphi$ ?  
Does  $\Sigma \models \varphi$  imply  $\Sigma \vdash \varphi$ ?



# Reflexivity / Premise

<b>Name</b>	<b><math>\vdash</math>-notation</b>	<b>inference notation</b>
Reflexivity, or Premise	$\Sigma, \alpha \vdash \alpha$	$\frac{\alpha}{\alpha}$

Given the formulas above the line, we can infer the formula below the line.

# An example using reflexivity

*Example.* Show that  $\{p, q\} \vdash p$ .

1.  $p$  Premise
  2.  $q$  Premise
  3.  $p$  Reflexivity: 1
- 
1.  $p$  Premise

For each symbol, the rules come in pairs.

- An “introduction rule” adds the symbol to the formula.
- An “elimination rule” removes the symbol from the formula.

# Rules for Conjunction

Name	$\vdash$ -notation	inference notation
$\wedge$ -introduction ( $\wedge i$ )	If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$ , then $\Sigma \vdash (\alpha \wedge \beta)$	$\frac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	$\vdash$ -notation	inference notation
$\wedge$ -elimination ( $\wedge e$ )	If $\Sigma \vdash (\alpha \wedge \beta)$ , then $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$	$\frac{(\alpha \wedge \beta)}{\alpha} \quad \frac{(\alpha \wedge \beta)}{\beta}$

# Example: Conjunction Rules

*Example.* Show that  $\{(p \wedge q)\} \vdash (q \wedge p)$ .

1.  $(p \wedge q)$  Premise
2.  $q$   $\wedge$ e: 1
3.  $p$   $\wedge$ e: 1
4.  $(q \wedge p)$   $\wedge$ i: 2, 3

## Example: Conjunction Rules (2)

*Example.* Show that  $\{(p \wedge q), r\} \vdash (q \wedge r)$ .

1.  $(p \wedge q)$  Premise
2.  $r$  Premise
3.  $q$   $\wedge$ e: 1
4.  $(q \wedge r)$   $\wedge$ i: 3, 2

# Rules for Implication: $\rightarrow$ e

Name	$\vdash$ -notation	inference notation
$\rightarrow$ -elimination ( $\rightarrow$ e) (modus ponens)	If $\Sigma \vdash (\alpha \rightarrow \beta)$ and $\Sigma \vdash \alpha$ , then $\Sigma \vdash \beta$	$\frac{(\alpha \rightarrow \beta) \quad \alpha}{\beta}$

# Rules for Implication: $\rightarrow$ i

Name	$\vdash$ -notation	inference notation
$\rightarrow$ -introduction ( $\rightarrow$ i)	If $\Sigma, \alpha \vdash \beta$ , then $\Sigma \vdash (\alpha \rightarrow \beta)$	$\frac{\boxed{\begin{array}{c} \alpha \\ \vdots \\ \beta \end{array}}}{(\alpha \rightarrow \beta)}$

The “box” denotes a sub-proof. In the sub-proof, we start by assuming that  $\alpha$  is true (a premise of the sub-proof), and we conclude that  $\beta$  is true.

Nothing inside the sub-proof may come out.

Outside of the sub-proof, we could only use the sub-proof as a whole.



## Example: Rule $\rightarrow$ i and sub-proofs

*Example.* Give a proof of  $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$ .

1.  $(p \rightarrow q)$  Premise
2.  $(q \rightarrow r)$  Premise
3.  $p$  Assumption
4.  $q$   $\rightarrow$ e: 1, 3
5.  $r$   $\rightarrow$ e: 2, 4
6.  $(p \rightarrow r)$   $\rightarrow$ i: 3–5

# Rules of Disjunction: $\vee$ i and $\vee$ e

Name	$\vdash$ -notation	inference notation
$\vee$ -introduction ( $\vee$ i)	If $\Sigma \vdash \alpha$ , then $\Sigma \vdash \alpha \vee \beta$ and $\Sigma \vdash \beta \vee \alpha$	$\frac{\alpha}{\alpha \vee \beta} \quad \frac{\alpha}{\beta \vee \alpha}$
$\vee$ -elimination ( $\vee$ e)	If $\Sigma, \alpha_1 \vdash \beta$ and $\Sigma, \alpha_2 \vdash \beta$ , then $\Sigma, \alpha_1 \vee \alpha_2 \vdash \beta$	$\frac{\alpha_1 \vee \alpha_2 \quad \boxed{\begin{array}{c} \alpha_1 \\ \vdots \\ \beta \end{array}} \quad \boxed{\begin{array}{c} \alpha_2 \\ \vdots \\ \beta \end{array}}}{\beta}$

$\vee$ e is also known as “proof by cases”.

# Example: Or-Introduction and -Elimination

*Example:* Show that  $\{p \vee q\} \vdash (p \rightarrow q) \vee (q \rightarrow p)$ .

1.	$p \vee q$	Premise
2.	$p$	Assumption
3.	$q$	Assumption
4.	$p$	Reflexivity: 2
5.	$q \rightarrow p$	$\rightarrow$ i: 3-4
6.	$(p \rightarrow q) \vee (q \rightarrow p)$	$\vee$ i: 5
7.	$q$	Assumption
8.	$p$	Assumption
9.	$q$	Reflexivity: 7
10.	$p \rightarrow q$	$\rightarrow$ i: 8-9
11.	$(p \rightarrow q) \vee (q \rightarrow p)$	$\vee$ i: 10
12.	$(p \rightarrow q) \vee (q \rightarrow p)$	$\vee$ e: 1, 2-6, 7-11

# Negation

We shall use the notation  $\perp$  to represent any contradiction. It may appear in proofs as if it were a formula.

Name	$\vdash$ -notation	inference notation
$\perp$ -introduction, or $\neg$ -elimination ( $\neg e$ )	$\Sigma, \alpha, (\neg\alpha) \vdash \perp$	$\frac{\alpha \quad (\neg\alpha)}{\perp}$

# Negation Introduction ( $\neg$ i)

If an assumption  $\alpha$  leads to a contradiction, then derive  $(\neg\alpha)$ .

Name	$\vdash$ -notation	inference notation
$\neg$ -introduction ( $\neg$ i)	If $\Sigma, \alpha \vdash \perp$ , then $\Sigma \vdash (\neg\alpha)$	$\frac{\boxed{\begin{array}{c} \alpha \\ \vdots \\ \perp \end{array}}}{(\neg\alpha)}$

# Example: Negation

*Example.* Show that  $\{\alpha \rightarrow (\neg\alpha)\} \vdash (\neg\alpha)$ .

1.  $\alpha \rightarrow (\neg\alpha)$  Premise
2.  $\alpha$  Assumption
3.  $(\neg\alpha)$   $\rightarrow$ e: 1, 2
4.  $\perp$   $\neg$ e: 2, 3
5.  $(\neg\alpha)$   $\neg$ i: 2-4

# The Last Two Basic Rules

Double-Negation Elimination:

Name	$\vdash$ -notation	inference notation
$\neg\neg$ -elimination ( $\neg\neg e$ )	If $\Sigma \vdash (\neg(\neg\alpha))$ , then $\Sigma \vdash \alpha$	$\frac{(\neg(\neg\alpha))}{\alpha}$

Contradiction Elimination:

Name	$\vdash$ -notation	inference notation
$\perp$ -elimination ( $\perp e$ )	If $\Sigma \vdash \perp$ , then $\Sigma \vdash \alpha$	$\frac{\perp}{\alpha}$

## Example: “*Modus tollens*”

*Modus tollens*:  $\{p \rightarrow q, (\neg q)\} \vdash (\neg p)$ .

- |    |                   |                       |
|----|-------------------|-----------------------|
| 1. | $p \rightarrow q$ | Premise               |
| 2. | $(\neg q)$        | Premise               |
| 3. | $p$               | Assumption            |
| 4. | $q$               | $\rightarrow$ e: 3, 1 |
| 5. | $\perp$           | $\neg$ e: 2, 4        |
| 6. | $(\neg p)$        | $\neg$ i: 3–5         |



# Strategies for natural deduction proofs

1. Work forward from the premises. What elimination rule can you apply to transform and/or simplify the premises?
2. Work backwards from the conclusion. What introduction rule can you use to produce the conclusion?
3. Use the structure of the formula to guide your proof.
4. If a direct proof doesn't work, try a proof by contradiction.

# Further Examples of Natural Deduction

*Example.* Show that  $\{p \rightarrow q\} \vdash (r \vee p) \rightarrow (r \vee q)$ .

1.	$p \rightarrow q$	Premise
2.	$r \vee p$	Assumption
3.	$r$	Assumption
4.	$r \vee q$	$\vee$ i: 3
5.	$p$	Assumption
6.	$q$	$\rightarrow$ e: 5, 1
7.	$r \vee q$	$\vee$ i: 6
8.	$r \vee q$	$\vee$ e: 2, 3–4, 5–7
9.	$(r \vee p) \rightarrow (r \vee q)$	$\rightarrow$ i: 2–8

## Some Common Derived Rules

Proof by contradiction (*reductio ad absurdum*):

if  $\Sigma, (\neg\alpha) \vdash \perp$ , then  $\Sigma \vdash \alpha$ .

The “Law of Excluded Middle” (*tertium non datur*):  $\vdash \alpha \vee (\neg\alpha)$ .

Double-Negation Introduction: if  $\Sigma \vdash \alpha$  then  $\Sigma \vdash (\neg(\neg\alpha))$ .

Try proving these yourself, as exercises.