Semantic Entailment and Natural Deduction

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Lecture 6

Learning goals

Semantic entailment

- Determine if a set of formulas is satisfiable.
- Define semantic entailment.
- Explain subtleties of semantic entailment.
- Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.

Natural deduction

- Describe rules of inference for natural deduction.
- Prove that a conclusion follows from a set of premises using natural deduction inference rules.

Proving arguments valid

Logic is the science of reasoning.

One goal of logic is to perform deductions — to infer that a conclusion is true based on a set of premises that we know to be true.

The process of logical deduction is formalized by the notion of semantic entailment.

Can we show that the conclusion semantically follows from the set of premises?

Semantic Entailment

Let Σ be a set of premises. Let φ be the conclusion.

Definition. A truth valuation t satisfies Σ (denoted $\Sigma^t = \mathtt{T}$) if and only if

for any formula α, if $\alpha \in \Sigma$, then $\alpha^t = \mathtt{T}$.

Definition. Σ semantically entails φ (denoted $\Sigma \vDash \varphi$) if and only if

for any truth valuation t, if $\Sigma^t = \mathtt{T}$, then $\varphi^t = \mathtt{T}$.

Prove an entailment

Consider the entailment $\Sigma \models \varphi$. To prove that the entailment holds, we need to consider

A) Every truth valuation t under which $\Sigma^t = T$.
B) Every truth valuation t under which $\Sigma^t = F$.
C) One truth valuation t under which $\Sigma^t = T$.
D) One truth valuation t under which $\Sigma^t = F$.

A semantic entailment does not hold

Let Σ be a set of premises. Let φ be the conclusion.

Definition. Σ semantically entails φ (denoted $\Sigma \vDash \varphi$) if and only if

for any truth valuation t, if $\Sigma^t = \mathtt{T}$, then $\varphi^t = \mathtt{T}$.

Definition. Σ does not entail φ (denoted $\Sigma \nvDash \varphi$ if and only if

there exists a truth valuation t such that $\Sigma^t = \mathtt{T}$ and $\varphi^t = \mathtt{F}$.

Disprove an entailment

Consider the entailment $\Sigma \models \varphi$. To prove that the entailment does NOT hold, we need to find

A) Every truth valuation t under which $\Sigma^t =$ *true* and $\varphi = \texttt{T}$.

B) Every truth valuation t under which $\Sigma^t =$ *true* and $\varphi = \texttt{F}$.

C) One truth valuation t under which $\Sigma^t =$ *true* and $\varphi = \texttt{T}$.

D) One truth valuation t under which $\Sigma^t =$ *true* and $\varphi = \texttt{F}$.

Proving/disproving an entailment using a truth table

Let $\Sigma = \{(\neg(p \wedge q)), (p \rightarrow q)\}$, $x = (\neg p)$, and $y = (p \leftrightarrow q)$. Based on the truth table, which of the following statements is true?

A) $\Sigma \models x$ and $\Sigma \models y$.

B) $\Sigma \models x$ and $\Sigma \not\models y$.

C) $\Sigma \not\models x$ and $\Sigma \models y$.

D) $\Sigma \not\models x$ and $\Sigma \not\models y$.

p	q	$(\neg(p \wedge q))$	$(p \rightarrow q)$	$x = (\neg p)$	$y = (p \leftrightarrow q)$
0	0	1	1	1	1
0	1	1	1	1	0
1	0	1	0	0	0
1	1	0	1	0	1

Proving/disproving an entailment using the definition

Exercise. Show that $\{(\neg(p \land q)), (p \rightarrow q)\} \vDash (\neg p)$.

Exercise. Show that $\{(\neg(p \land q)), (p \rightarrow q)\} \nvDash (p \leftrightarrow q)$.

Subtleties of entailment

Consider the entailment $\Sigma \vDash \varphi$.

Does the entailment hold under each of the following conditions?

1. Σ is the empty set.
2. Σ is not satisfiable.
3. φ is a tautology.
4. φ is a contradiction.

Propositional Logic:

Natural Deduction

Learning goals

Natural deduction in propositional logic

- Describe rules of inference for natural deduction.
- Prove a conclusion from given premises using natural deduction inference rules.
- Describe strategies for applying each inference rule when proving a conclusion formula using natural deduction.

The Natural Deduction Proof System

We will consider a proof system called Natural Deduction.

- It closely follows how people (mathematicians, at least) normally make formal arguments.
- It extends easily to more-powerful forms of logic.

Why would you want to study natural deduction proofs?

- It is impressive to be able to write proofs with nested boxes and mysterious symbols as justifications.
- Be able to prove or disprove that Superman exists (on Tuesday).
- Be able to prove or disprove that the onnagata are correct to insist that males should play female characters in Japanese kabuki theatres.
- To realize that writing proofs and problem solving in general is both a creative and a scientific endeavour.
- To develop problem solving strategies that can be used in many other situations.

A proof is syntactic

First, we think about proofs in a purely syntactic way.

A proof

- starts with a set of premises,
- transforms the premises based on a set of inference rules (by pattern matching),
- and reaches a conclusion.

We write

$$\Sigma \vdash_{ND} \varphi \quad \text{or simply} \quad \Sigma \vdash \varphi$$

if we can find such a proof that starts with a set of premises Σ and ends with the conclusion φ.

Goal is to show semantic entailment

Next, we think about connecting proofs to semantic entailment.

We will answer these questions:

- (Soundness) Does every proof establish a semantic entailment?
  If I can find a proof from Σ to φ, can I conclude that Σ semantically entails φ?
  Does $\Sigma \vdash \varphi$ imply $\Sigma \vDash \varphi$?
- (Completeness) For every semantic entailment, can I find a proof for it?
  If I know that Σ semantically entails φ, can I find a proof from Σ to φ?
  Does $\Sigma \vDash \varphi$ imply $\Sigma \vdash \varphi$?

Reflexivity / Premise

Name	**$\vdash$-notation**	**inference notation**
Reflexivity, or Premise	$\Sigma, \alpha \vdash \alpha$	$\frac{\alpha}{\alpha}$

Given the formulas above the line, we can infer the formula below the line.

An example using reflexivity

Example. Show that $\{p, q\} \vdash p$.

1.	p	Premise
2.	q	Premise
3.	p	Reflexivity: 1

1.	p	Premise

For each symbol, the rules come in pairs.

- An “introduction rule” adds the symbol to the formula.
- An “elimination rule” removes the symbol from the formula.

Name	$\vdash$-notation	inference notation
$\wedge$-introduction ($\wedge$i)	If $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$, then $\Sigma \vdash (\alpha \wedge \beta)$	$\dfrac{\alpha \quad \beta}{(\alpha \wedge \beta)}$

Name	$\vdash$-notation	inference notation
$\wedge$-elimination ($\wedge$e)	If $\Sigma \vdash (\alpha \wedge \beta)$, then $\Sigma \vdash \alpha$ and $\Sigma \vdash \beta$	$\dfrac{(\alpha \wedge \beta)}{\alpha} \quad \dfrac{(\alpha \wedge \beta)}{\beta}$

Example: Conjunction Rules

Example. Show that $\{(p \wedge q)\} \vdash (q \wedge p)$.

1.	$(p \wedge q)$	Premise
2.	q	$\wedge$e: 1
3.	p	$\wedge$e: 1
4.	$(q \wedge p)$	$\wedge$i: 2, 3

Example: Conjunction Rules (2)

Example. Show that $\{(p \wedge q), r\} \vdash (q \wedge r)$.

1.	$(p \wedge q)$	Premise
2.	r	Premise
3.	q	$\wedge$e: 1
4.	$(q \wedge r)$	$\wedge$i: 3, 2

Rules for Implication: $\to$e

Name	**$\vdash$-notation**	**inference notation**
$\to$-elimination ($\to$e) (modus ponens)	If $\Sigma \vdash (\alpha \to \beta)$ and $\Sigma \vdash \alpha$, then $\Sigma \vdash \beta$	$\dfrac{(\alpha \to \beta) \quad \alpha}{\beta}$

Rules for Implication: $\rightarrow$i

Name	**$\vdash$-notation**	**inference notation**
$\rightarrow$-introduction ($\rightarrow$i)	If $\Sigma, \alpha \vdash \beta$, then $\Sigma \vdash (\alpha \rightarrow \beta)$	$\dfrac{\boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}}}{(\alpha \rightarrow \beta)}$

The "box" denotes a sub-proof. In the sub-proof, we starts by assuming that α is true (a premise of the sub-proof), and we conclude that β is true.

Nothing inside the sub-proof may come out.

Outside of the sub-proof, we could only use the sub-proof as a whole.

Example: Rule $\rightarrow$i and sub-proofs

Example. Give a proof of $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$.

1.	$(p \rightarrow q)$	Premise
2.	$(q \rightarrow r)$	Premise
3.	p	Assumption
4.	q	$\rightarrow$e: 1, 3
5.	r	$\rightarrow$e: 2, 4
6.	$(p \rightarrow r)$	$\rightarrow$i: 3–5

Name	**$\vdash$-notation**	**inference notation**
$\vee$-introduction ($\vee$i)	If $\Sigma \vdash \alpha$, then $\Sigma \vdash \alpha \vee \beta$ and $\Sigma \vdash \beta \vee \alpha$	$\dfrac{\alpha}{\alpha \vee \beta}$ $\dfrac{\alpha}{\beta \vee \alpha}$
$\vee$-elimination ($\vee$e)	If $\Sigma, \alpha_1 \vdash \beta$ and $\Sigma, \alpha_2 \vdash \beta$, then $\Sigma, \alpha_1 \vee \alpha_2 \vdash \beta$	$\dfrac{\alpha_1 \vee \alpha_2 \quad \boxed{\alpha_1 \; \vdots \; \beta} \quad \boxed{\alpha_2 \; \vdots \; \beta}}{\beta}$

$\vee$e is also known as "proof by cases".

Example: Or-Introduction and -Elimination

Example: Show that $\{p \vee q\} \vdash (p \to q) \vee (q \to p)$.

1.	$p \vee q$	Premise
2.	p	Assumption
3.	q	Assumption
4.	p	Reflexivity: 2
5.	$q \to p$	$\to$i: 3–4
6.	$(p \to q) \vee (q \to p)$	$\vee$i: 5
7.	q	Assumption
8.	p	Assumption
9.	q	Reflexivity: 7
10.	$p \to q$	$\to$i: 8–9
11.	$(p \to q) \vee (q \to p)$	$\vee$i: 10
12.	$(p \to q) \vee (q \to p)$	$\vee$e: 1, 2–6, 7–11

We shall use the notation $\perp$ to represent any contradiction. It may appear in proofs as if it were a formula.

Name	$\vdash$**-notation**	**inference notation**
$\perp$-introduction, or $\neg$-elimination ($\neg$e)	$\Sigma,\ \alpha,\ (\neg\alpha) \vdash \perp$	$\dfrac{\alpha \quad (\neg\alpha)}{\perp}$

Negation Introduction ($\neg$i)

If an assumption α leads to a contradiction, then derive $(\neg\alpha)$.

Name	**$\vdash$-notation**	**inference notation**
$\neg$-introduction ($\neg$i)	If $\Sigma, \alpha \vdash \bot$, then $\Sigma \vdash (\neg\alpha)$	$\dfrac{\boxed{\begin{array}{c}\alpha \\ \vdots \\ \bot\end{array}}}{(\neg\alpha)}$

Example: Negation

Example. Show that $\{\alpha \rightarrow (\neg\alpha)\} \vdash (\neg\alpha)$.

1.	$\alpha \rightarrow (\neg\alpha)$	Premise
2.	α	Assumption
3.	$(\neg\alpha)$	$\rightarrow$e: 1, 2
4.	$\bot$	$\neg$e: 2, 3
5.	$(\neg\alpha)$	$\neg$i: 2–4

Double-Negation Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\neg\neg$-elimination ($\neg\neg$e)	If $\Sigma \vdash (\neg(\neg\alpha))$, then $\Sigma \vdash \alpha$	$\dfrac{(\neg(\neg\alpha))}{\alpha}$

Contradiction Elimination:

Name	**$\vdash$-notation**	**inference notation**
$\bot$-elimination ($\bot$e)	If $\Sigma \vdash \bot$, then $\Sigma \vdash \alpha$	$\dfrac{\bot}{\alpha}$

Example: "*Modus tollens*"

Modus tollens: $\{p \to q, (\neg q)\} \vdash (\neg p)$.

1.	$p \to q$	Premise
2.	$(\neg q)$	Premise
3.	p	Assumption
4.	q	$\to$e: 3, 1
5.	$\bot$	$\neg$e: 2, 4
6.	$(\neg p)$	$\neg$i: 3–5

Strategies for natural deduction proofs

1. Work forward from the premises. What elimination rule can you apply to transform and/or simplify the premises?
2. Work backwards from the conclusion. What introduction rule can you use to produce the conclusion?
3. Use the structure of the formula to guide your proof.
4. If a direct proof doesn't work, try a proof by contradiction.

Further Examples of Natural Deduction

Example. Show that $\{p \to q\} \vdash (r \lor p) \to (r \lor q)$.

1.	$p \to q$	Premise
2.	$r \lor p$	Assumption
3.	r	Assumption
4.	$r \lor q$	$\lor$i: 3
5.	p	Assumption
6.	q	$\to$e: 5, 1
7.	$r \lor q$	$\lor$i: 6
8.	$r \lor q$	$\lor$e: 2, 3–4, 5–7
9.	$(r \lor p) \to (r \lor q)$	$\to$i: 2–8

Some Common Derived Rules

Proof by contradiction (*reductio ad absurdum*):

$$\text{if } \Sigma, (\neg\alpha) \vdash \bot, \text{ then } \Sigma \vdash \alpha.$$

The "Law of Excluded Middle" (*tertiam non datur*): $\vdash \alpha \vee (\neg\alpha)$.

Double-Negation Introduction: if $\Sigma \vdash \alpha$ then $\Sigma \vdash (\neg(\neg\alpha))$.

Try proving these yourself, as exercises.