Propositional Logic: Semantics

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Lecture 5

Announcements

Clickers

- What are they for? Active learning, engagement
- Why do you ask us to answer twice? Peer instruction
- If you choose to be here, please participate!

Learning goals

By the end of this lecture, you should be able to:

(Logical equivalence)

- Prove that two formulas are logically equivalent using truth tables.
- Prove that two formulas are logically equivalent using logical identities.
- Translate a condition in a block of code into a propositional formula. Simplify an if statement. Determine whether a piece of code is live or dead.

(Adequate set of connectives)

- Prove that a set of connectives is adequate.
- Prove that a set of connectives is not adequate.

Definition of logical equivalence

Two formulas α and β are logically equivalent, denoted $\alpha \equiv \beta$ if and only if

- $\alpha^t = \beta^t$, for every valuation t.
- α and β have the same final column in their truth tables.
- $(\alpha \leftrightarrow \beta)$ is a tautology.

Why do we care about logical equivalence?

- Do these two formulas have the same meaning?
- Do these two circuits behave the same way?
- Do these two pieces of code fragments behave the same way?

You already know one way of proving logical equivalent. What is it?

Theorem: $(((\neg p) \wedge q) \vee p) \equiv (p \vee q)$.

Logical Identities

Commutativity

$$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$$
$$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$$

Associativity

$$\big(\alpha \wedge (\beta \wedge \gamma)\big) \equiv \big((\alpha \wedge \beta) \wedge \gamma\big)$$
$$\big(\alpha \vee (\beta \vee \gamma)\big) \equiv \big((\alpha \vee \beta) \vee \gamma\big)$$

Distributivity

$$\big(\alpha \vee (\beta \wedge \gamma)\big) \equiv \big((\alpha \vee \beta) \wedge (\alpha \vee \gamma)\big)$$
$$\big(\alpha \wedge (\beta \vee \gamma)\big) \equiv \big((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\big)$$

Idempotence

$$(\alpha \vee \alpha) \equiv \alpha$$
$$(\alpha \wedge \alpha) \equiv \alpha$$

Double Negation

$$\big(\neg(\neg\alpha)\big) \equiv \alpha$$

De Morgan's Laws

$$\big(\neg(\alpha \wedge \beta)\big) \equiv \big((\neg\alpha) \vee (\neg\beta)\big)$$
$$\big(\neg(\alpha \vee \beta)\big) \equiv \big((\neg\alpha) \wedge (\neg\beta)\big)$$

Logical Identities, cont'd

Simplification I (Absorbtion)

$$(\alpha \land \mathtt{T}) \equiv \alpha$$
$$(\alpha \lor \mathtt{T}) \equiv \mathtt{T}$$
$$(\alpha \land \mathtt{F}) \equiv \mathtt{F}$$
$$(\alpha \lor \mathtt{F}) \equiv \alpha$$

Simplification II

$$(\alpha \lor (\alpha \land \beta)) \equiv \alpha$$
$$(\alpha \land (\alpha \lor \beta)) \equiv \alpha$$

Implication

$$(\alpha \to \beta) \equiv ((\neg\alpha) \lor \beta)$$

Contrapositive

$$(\alpha \to \beta) \equiv ((\neg\beta) \to (\neg\alpha))$$

Equivalence

$$(\alpha \leftrightarrow \beta) \equiv ((\alpha \to \beta) \land (\beta \to \alpha))$$

Excluded Middle

$$(\alpha \lor (\neg\alpha)) \equiv \mathtt{T}$$

Contradiction

$$(\alpha \land (\neg\alpha)) \equiv \mathtt{F}$$

Proving logical equivalence

"If it is sunny, I will play golf, provided that I am relaxed."

s: it is sunny. g: I will play golf. r: I am relaxed.

Three possible translations:

1. $(s \rightarrow (r \rightarrow g))$
2. $(r \rightarrow (s \rightarrow g))$
3. $((s \wedge r) \rightarrow g)$

Prove that all three translations are logically equivalent.

Proof.

... to be filled in ... $\square$

Strategies for improving logical equivalence

- Try getting rid of $\to$ and $\leftrightarrow$.
- Try moving negations inward using De Morgan's law. $(\neg(p \vee q)) \equiv (\neg p) \wedge (\neg q)$.
- Work from the more complex side first.
- Switch to different strategies/sides when you get stuck.
- In the end, write the proof in clean "one-side-to-the-other" form and double-check steps.

Proving non-equivalence

"If it snows then I won't go to class but I will do my assignment."

s: it snows. c: I will go to class. a: I will do my assignment.

2 possible translations:

1. $((s \to (\neg c)) \wedge a)$
2. $(s \to ((\neg c) \wedge a))$

Theorem: The two translations are NOT logically equivalent.

Proving non-equivalence

"If it snows then I won't go to class but I will do my assignment."

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2 possible translations:

1. $((s \to (\neg c)) \land a)$
2. $(s \to ((\neg c) \land a))$

Theorem: The two translations are NOT logically equivalent.

Which valuation t can we use to prove this theorem?

(a) $t(s) = \mathtt{F}$, $t(c) = \mathtt{T}$, $t(a) = \mathtt{F}$
(b) $t(s) = \mathtt{F}$, $t(c) = \mathtt{F}$, $t(a) = \mathtt{F}$
(c) $t(s) = \mathtt{T}$, $t(c) = \mathtt{F}$, $t(a) = \mathtt{T}$
(d) Two of (a), (b), and (c).
(e) All of (a), (b), and (c).

A piece of pseudo code

if `( (input > 0) or (not output) ) {`
    if `( not (output and (queuelength < 100) ) ) {`
        P_1
    `}` **else if** `( output and (not (queuelength < 100)) ) {`
        P_2
    `}` **else** `{` P_3 `}`
`}` **else** `{` P_4 `}`

When does each piece of code get executed?

Let i: `input` > 0,
u: `output`,
q: `queuelength` < 100.

```
if ( i or (not u) ) {
    if ( not (u and q) ) {
        P1
    } else if ( u and (not q) ) {
        P2
    } else {
        P3
    }
} else {
    P4
}
```

When does each piece of code get executed?

i	u	q	Code block
T	T	T	P_3
T	T	F	P_1
T	F	T	P_1
T	F	F	P_1
F	T	T	P_4
F	T	F	P_4
F	F	T	P_1
F	F	F	P_1

Finding Dead Code

Prove that P_2 is dead code. That is, the conditions leading to P_2 is logically equivalent to `F`.

Proof.

... to be filled in ... $\square$

What is P_3 executed?

Prove that P_3 is live code. Specifically, prove that P_3 is executed if and only if i, u and q are all true.

Proof.

... to be filled in ... $\square$

When is P_4 executed?

Prove that P_4 is live code. Specifically, prove that P_4 is executed if and only if i is false and u is true.

Proof.

... to be filled in ... $\square$

Two equivalent pieces of code

We have shown that the following two pieces of code are equivalent.

Fragment 1:

```
if (  i or (not u) ) {
    if (   not (u and q) ) {
        P1
    }
    else if (  u and (not q) ) {
        P2
    }
    else {
        P3
    }
}
else {
    P4
}
```

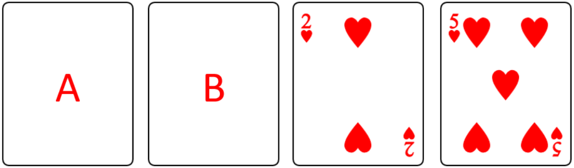
Fragment 2:

```
if (  (i and u) and q ) {
    P3
}
else if (  (not i) and u ) {
    P4
}
else {
    P1
}
```

A logic puzzle

Each of the four cards has a letter on one side and a natural number on the other side. Which cards do you have to turn over to determine whether or not the following claim is true?

Claim: For each card, if the card has a vowel on one side, then it has an even number on the other side.

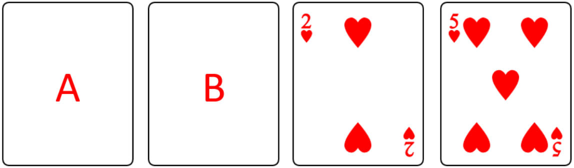


A logic puzzle: The solution

Each of the four cards has a letter on one side and a natural number on the other side. Which cards do you have to turn over to determine whether or not the following claim is true?

Claim: For each card, if the card has a vowel on one side, then it has an even number on the other side.

Solution: You need to turn over the first and the fourth cards from the left. If a card has a vowel on one side, you need to check that it has an even number on the other side. If a card has an odd number on one side, you need to check that it has a non-vowel on the other side (this is the contrapositive of the given claim).



Another logic puzzle

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

You meet three inhabitants: Alice, Rex and Bob.

Alice says, "Rex is a knave."

Rex says, "it's false that Bob is a knave."

Bob claims, "I am a knight or Alice is a knight."

Can you determine who is a knight and who is a knave?

Adequate sets of connectives

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(Adequate set of connectives)

- Prove that a set of connectives is adequate.
- Prove that a set of connectives is not adequate.

Some questions first

- We started propositional logic by learning these connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.
- Why did we learn these connectives?
- Using these connectives, can we express every propositional logic formula that we ever want to write?
- Are there any connectives in this set that are not necessary?
- Are there other connectives that we could define and use? Is there another set of connectives that we should have studied instead?

Some answers

Is every connective we learned necessary?

Some answers

Is every connective we learned necessary?

Nope!

Recall that $(x \to y) \equiv ((\neg x) \vee y)$. We don't need $\to$ at all. (We say that $\to$ is definable in terms of $\neg$ and $\vee$.)

Some answers

Is every connective we learned necessary?

Nope!

Recall that $(x \to y) \equiv ((\neg x) \lor y)$. We don't need $\to$ at all. (We say that $\to$ is definable in terms of $\neg$ and $\lor$.)

Are there other connectives that we could define and use?

Yep! Let's take a look.

Adequate Sets of Connectives

Which set of connectives is sufficient to express every possible propositional formula?

Definition

A set of connectives is adequate for propositional logic if and only if it can express every possible propositional formula.

In other words, any other connective is definable in terms of the ones in an adequate set.

An adequate set to start with

Theorem 1: $\{\wedge, \vee, \neg\}$ is an adequate set of connectives.

Proof Sketch.

For every propositional formula, we can write down a truth table for it.

We can convert every truth table to a proposition formula, which only contains $\wedge$, $\vee$, and $\neg$. (How?) $\square$

A reduction problem

Theorem 2: Each of $\{\wedge, \neg\}$, $\{\vee, \neg\}$, and $\{\rightarrow, \neg\}$ is adequate.

We will prove that $\{\wedge, \neg\}$ is adequate. The proofs for the other two sets are similar.

Proof Sketch.

By Theorem 1, the set $\{\wedge, \vee, \neg\}$ is adequate.

To prove that $\{\wedge, \neg\}$ is adequate, we need to show that $\vee$ is definable in terms of $\wedge$ and $\neg$. (How?) $\square$

A non-adequate set

Theorem 3. The set $\{\wedge, \vee\}$ is NOT an adequate set of connectives.

Consider a formula, which uses only $\wedge$ and $\vee$ as connectives. Consider a valuation in which every variable is true. What is the truth value of the formula under this valuation?

(a) True.

(b) False.

(c) Depends on the formula.

A non-adequate set

Theorem 3. The set $\{\wedge, \vee\}$ is *not* an adequate set of connectives.

Proof Sketch.

First we prove the following lemma using structural induction.

Lemma: Consider a valuation in which every variable is true. A formula is true if it only uses $\wedge$ and $\vee$ as its connectives.

By using this lemma, we can prove that $\neg$ is NOT definable using only $\wedge$ and $\vee$. Thus, $\{\wedge, \vee\}$ is *not* an adequate set of connectives. $\square$