Propositional Logic: Semantics

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Lecture 4

Announcements

The roadmap of propositional logic

FCC spectrum auction — an application of propositional logic

Goal is to re-purpose radio spectrum. 2 auctions: one to buy back spectrum from broadcasters, and the other to sell spectrum to telecoms.

A computational problem in the buy back auction: If I pay these broadcasters to go off air, could I repackage the spectrums and sell to telecoms? Could I lower your price and still manage to get useful spectrums to sell to telecoms?

The problem comes down to, how many satisfiability problems can I solve in a very short amount of time? (determine that a formula is satisfiable/unsatisfiable.)

Talk by Kevin Leyton-Brown
`https://www.youtube.com/watch?v=u1-jJOivP7O`

Learning goals

By the end of this lecture, you should be able to

(Truth valuation, truth table, and valuation tree)

- Define a truth valuation.
- Determine the truth value of a formula given a truth valuation.
- Give a truth valuation under which a formula is true or false.
- Draw a truth table given a formula.
- Draw the valuation tree given a formula.

By the end of this lecture, you should be able to

(Properties of formulas)

- Define tautology, contradiction, and satisfiable formula.
- Determine if a given formula is a tautology, a contradiction, and/or a satisfiable formula.

The meaning of well-formed formulas

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A propositional variable has no intrinsic meaning; it gets a meaning via a valuation.

A truth valuation assigns true/false to every propositional variable.

What is a truth valuation, intuitively?

Have you watched the TV series called Fringe?

A truth valuation defines a parallel universe.

In each parallel universe, the truth values of each propositional may be different from their truth values in our universe.

Our universe: The sky is blue and pigs do not fly.

Parallel universe 1: The sky is red, and pigs do not fly.

Parallel universe 2: The sky is blue, and pigs fly.

Definition of a truth valuation

A (truth) valuation is a function $t : \mathcal{P} \mapsto \{\mathtt{F}, \mathtt{T}\}$ from the set of all proposition variables $\mathcal{P}$ to the set $\{\mathtt{F}, \mathtt{T}\}$.

Notation:

- For a propositional variable p and a truth valuation t, both $t(p)$ and p^t denote the value of p under t.
- For a well-formed formula φ and a truth valuation t, φ^t denotes the truth value of φ under t.

Truth tables for connectives

Every line in a truth table corresponds to a truth valuation.

The unary connective $\neg$:

α	$(\neg\alpha)$
T	F
F	T

The binary connectives $\wedge$, $\vee$, $\to$, and $\leftrightarrow$:

α	β	$(\alpha \wedge \beta)$	$(\alpha \vee \beta)$	$(\alpha \to \beta)$	$(\alpha \leftrightarrow \beta)$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

A structural induction example

Theorem: Fix a truth valuation t. The truth value of every formula α is in $\{\mathtt{F}, \mathtt{T}\}$.

Prove this yourself.

Evaluating the truth value of a formula

Consider the formula $\big((p \to (\neg q)) \to (q \vee (\neg p))\big)$.

Exercises:

1. Determine the truth value of this formula under the truth valuation t_1: $t_1(p) = \mathtt{T}$, $t_1(q) = \mathtt{T}$.
2. Give a truth valuation t_2 under which this formula is false.

p	q	$(p \to (\neg q))$	$(q \vee (\neg p))$	$((p \to (\neg q)) \to (q \vee (\neg p)))$
F	F	T	T	T
F	T	T	T	T
T	F	T	F	F
T	T	F	T	T

Understanding the disjunction and the biconditional

α	β	$(\alpha \vee \beta)$	Exclusive OR	Biconditional
F	F	F	F	T
F	T	T	T	F
T	F	T	T	F
T	T	T	F	T

- What is the difference between an inclusive OR (the disjunction) and an exclusive OR?
- What is the relationship between the exclusive OR and the biconditional?

Understanding an implication

Consider the proposition "If Alice is rich, she pays your tuition."

Assume that the above proposition is true.

1. If Alice is rich, does she pay your tuition?
   (a) Yes (b) No (c) Maybe
2. If Alice is not rich, does she pay your tuition?
   (a) Yes (b) No (c) Maybe

Understanding an implication

To model the implication, let's define the following propositional variables.

r: Alice is rich.

t: Alice pays for your tuition.

The implication becomes $(r \to t)$ with the following truth table.

r	t	$(r \to t)$
F	F	T
F	T	T
T	F	F
T	T	T

Proving that an implication is true

Consider the implication “If Alice is rich, she pays your tuition.”

1. When is the implication false? (Think of it as a promise that I made to you. How can you prove that I have broken my promise?)
2. How do you prove that the implication is false?
3. How do you prove that the implication is true?

Tautology, Contradiction, Satisfiable

A formula α is a *tautology* if and only if for every truth valuation t, $\alpha^t = \mathtt{T}$.

A formula α is a *contradiction* if and only if for every truth valuation t, $\alpha^t = \mathtt{F}$.

A formula α is *satisfiable* if and only if there exists a truth valuation t such that $\alpha^t = \mathtt{T}$.

Relationships among the properties

Divide the set of all formulas into 3 mutually exclusive and exhaustive sets.

1. Tautology: Each formula is
   true in EVERY/AT LEAST ONE/NO row of its truth table, and
   false in EVERY/AT LEAST ONE/NO row of its truth table.
2. Satisfiable but not a tautology: Each formula is
   true in EVERY/AT LEAST ONE/NO row of its truth table, and
   false in EVERY/AT LEAST ONE/NO row of its truth table.
3. Contradiction: Each formula is
   true in EVERY/AT LEAST ONE/NO row of its truth table, and
   false in EVERY/AT LEAST ONE/NO row of its truth table.

Is a formula a tautology, contradiction, and/or satisfiable?

Three approaches:

- Truth table
- Valuation tree
- (Ad-hoc) reasoning

Simplifying a formula

Rather than filling out an entire truth table, we can simplify a formula in many situations:

$$
\begin{array}{ccc}
 & & p \rightarrow \mathtt{T} \equiv ? \\
p \wedge \mathtt{T} \equiv ? & p \vee \mathtt{T} \equiv ? & p \rightarrow \mathtt{F} \equiv ? \\
p \wedge \mathtt{F} \equiv ? & p \vee \mathtt{F} \equiv ? & \mathtt{T} \rightarrow p \equiv ? \\
p \wedge p \equiv ? & p \vee p \equiv ? & \mathtt{F} \rightarrow p \equiv ? \\
 & & p \rightarrow p \equiv ?
\end{array}
$$

We can evaluate a formula by constructing a valuation tree using these rules.

Example of a valuation tree

Example. Show that $\Big(\big(((p \wedge q) \to (\neg r)) \wedge (p \to q)\big) \to (p \to (\neg r))\Big)$ is a tautology by using a valuation tree.

Case 1: Suppose $t(p) = \mathtt{T}$. The formula becomes $\Big(\big((q \to (\neg r)) \wedge q\big) \to (\neg r)\Big)$.

If $t(q) = \mathtt{T}$, the formula is $\mathtt{T}$ (Check!).

If $t(q) = \mathtt{F}$, the formula is $\mathtt{T}$ (Check!).

Case 2: Suppose $t(p) = \mathtt{F}$. The formula is $\mathtt{T}$. (Check!).

The formula is true for every valuation and is a tautology.

Note: We never had to consider the truth value of r in our analysis.

One example of ad-hoc reasoning

I found a valuation for which the formula is true. Does the formula have each property below?

▪ Tautology	YES	NO	MAYBE
▪ Contradiction	YES	NO	MAYBE
▪ Satisfiable	YES	NO	MAYBE

I found a valuation for which the formula is false. Does the formula have each property below?

▪ Tautology	YES	NO	MAYBE
▪ Contradiction	YES	NO	MAYBE
▪ Satisfiable	YES	NO	MAYBE

Determine if each of the following formulas is a tautology, a contradiction, and/or satisfiable.

1. $\Big(\big(((p \wedge q) \rightarrow r) \wedge (p \rightarrow q)\big) \rightarrow (p \rightarrow r)\Big)$
2. $(p \vee q) \leftrightarrow ((p \wedge (\neg q) \vee ((\neg p) \wedge q))$
3. $(p \wedge (\neg p))$