Propositional Logic: Syntax and Structural Induction

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Lecture 2

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Outline

Admin stuff

Learning goals

By the end of the lecture, you should be able to
(Well-formed formulas)

- Describe the three types of symbols in propositional logic.
- Give the inductive definition of well-formed formulas.
- Write the parse tree for a well-formed formula.
- Determine and justify whether a given formula is well formed.

(Structural induction)

- Prove properties of well-formed propositional formulas using structural induction.
- Prove properties of a recursively defined concept using structural induction.

Propositional logic symbols

Three types of symbols in propositional logic:

- Propositional variables: p, q, r, p_1, etc.
- Connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.
- Punctuation: (and).

Expressions

An expression is a string of symbols.
Examples:

- α: $(\neg)()\vee pq\rightarrow$
- β: $a \vee b \wedge c$
- γ: $((a \rightarrow b) \vee c)$

However, an expression is useful to us if and only if it has a unique meaning.

Definition of well-formed formulas

Let $\mathcal{P}$ be a set of propositional variables. We define the set of well-formed formulas over $\mathcal{P}$ inductively as follows.

1. A propositional variable in $\mathcal{P}$ is well-formed.
2. If α is well-formed, then $(\neg\alpha)$ is well-formed.
3. If α and β are well-formed, then each of $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \to \beta), (\alpha \leftrightarrow \beta)$ is well-formed.

CQ Are these formulas well-formed?

The parse tree of a well-formed formula

For a complex formula, its parse tree makes the structure of the formula explicit.

Draw the parse tree of the following formulas.

1. $((a \vee b) \wedge (\neg(a \wedge b)))$
2. $(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$.

Unique readability of well-formed formulas

Does every well-formed formula have a unique meaning? Yes.

Theorem: There is a unique way to construct each well-formed formula.

Properties of well-formed formulas

We may want to prove other properties of well-formed formulas.

- Every well-formed formula has at least one propositional variable.
- Every well-formed formula has an equal number of opening and closing brackets.
- Every proper prefix of a well-formed formula has more opening brackets than closing brackets.
- There is a unique way to construct every well-formed formula.

Why should you care?

Learning goals on structural induction:

- Prove properties of well-formed propositional formulas using structural induction.
- Prove properties of a recursively defined concept using structural induction.

Learning goals for future courses:

- Prove the space and time efficiency of recursive algorithms using induction.

Properties of well-formed formulas

Theorem: For every well-formed propositional formula φ, $P(\varphi)$ is true.

Induction over natural numbers

Let the natural numbers start from 0. Let P be some property. We want to prove that every natural number has property P.
Theorem: $P(0)$, $P(1)$, $P(2)$, ..., are all true.

Proof.

Base case: Prove $P(0)$.
Induction step: Consider an arbitrary $k \geq 0$. Assume that $P(k)$ is true. Prove that $P(k+1)$ is true.
By the principle of mathematical induction, $P(n)$ is true for $n = 1, 2, 3, \ldots$. □

Structural induction on well-formed formulas

Step 1: Identify the recursive structure in the problem.

Theorem: Every well-formed formula has an equal number of opening and closing brackets.

Notes:

- The "well-formed formula" is the recursive structure.
- "Has an equal number of opening and closing brackets" is the property of well-formed formulas.

Structural induction on well-formed formulas

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does the object reference itself?

Let $\mathcal{P}$ be a set of propositional variables. We define the set of well-formed formulas over $\mathcal{P}$ inductively as follows.

1. A propositional variable in $\mathcal{P}$ is well-formed.
2. If α is well-formed, then $(\neg\alpha)$ is well-formed.
3. If α and β are well-formed, then each of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$ is well-formed.

Structural induction on well-formed formulas

Let $\mathcal{P}$ be a set of propositional variables. We define the set of well-formed formulas over $\mathcal{P}$ inductively as follows.

1. A propositional variable in $\mathcal{P}$ is well-formed.
2. If α is well-formed, then $(\neg\alpha)$ is well-formed.
3. If α and β are well-formed, then each of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$ is well-formed.

Which of the three cases have recursive appearances of well-formed formulas?

(A) 2

(B) 3

(C) 2, 3

(D) 1, 2, 3

(E) None of the above

Structural induction on well-formed formulas

Step 2: Identify each recursive appearance of the structure inside its definition. (A recursive structure is self-referential. Where in the definition of the object does the object reference itself?

Let $\mathcal{P}$ be a set of propositional variables. We define the set of well-formed formulas over $\mathcal{P}$ inductively as follows.

1. A propositional variable in $\mathcal{P}$ is well-formed. (Non-recursive)
2. If α is well-formed, then $(\neg\alpha)$ is well-formed. (Recursive)
3. If α and β are well-formed, then each of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$ is well-formed. (Recursive)

Structural induction on well-formed formulas

Step 3: The cases without recursive appearances are the “base cases”. Those with recursive appearances are the “inductive cases”.

Let $\mathcal{P}$ be a set of propositional variables. We define the set of well-formed formulas over $\mathcal{P}$ inductively as follows.

1. A propositional variable in $\mathcal{P}$ is well-formed. (Base case)
2. If α is well-formed, then $(\neg\alpha)$ is well-formed. (Inductive case)
3. If α and β are well-formed, then each of $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, and $(\alpha \leftrightarrow \beta)$ is well-formed. (Inductive case)

A structural induction template for well-formed formulas

Theorem: For every well-formed formula φ, $P(\varphi)$ holds.

Proof by structural induction:

Base case: φ is a propositional variable q. Prove that $P(q)$ holds.

Induction step:

Case 1: φ is $(\neg a)$, where a is well-formed.
Induction hypothesis: Assume that $P(a)$ holds.
We need to prove that $P((\neg a))$ holds.

Case 2: φ is $(a * b)$ where a and b are well-formed and $*$ is a binary connective.
Induction hypothesis: Assume that $P(a)$ and $P(b)$ hold.
We need to prove that $P((a * b))$ holds.

By the principle of structural induction, $P(\varphi)$ holds for every well-formed formula φ. QED

Review questions about the structural induction template

1. Why is the definition of a well-formed formula recursive?
2. To prove a property of well-formed formulas using structural induction, how many base cases and inductive cases are there in the proof?
3. In the base case, how do we prove the theorem? Does the proof rely on any additional assumption about the formula?
4. In an inductive case, how do we prove the theorem? Does the proof rely on any additional assumption about the formula?

Structural induction problems

Problem 1: Every well-formed formula has at least one propositional variable.
Problem 2: Every well-formed formula has an equal number of opening and closing brackets.
Problem 3: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

Balanced brackets in a well-formed formula

Problem: Every well-formed formula has an equal number of opening and closing brackets.
What is the induction hypothesis in case 2 of the induction step?

(a) α and β are both well-formed formulas.

(b) Each of α and β has an equal number of opening and closing brackets.

(c) $(\alpha \wedge \beta)$ has an equal number of opening and closing brackets.

In case 2 of the induction step, on which line did we apply the induction hypothesis?

(a) $\mathsf{op}((\alpha * \beta)) = 1 + \mathsf{op}(\alpha) + \mathsf{op}(\beta)$

(b) $1 + \mathsf{op}(\alpha) + \mathsf{op}(\beta) = 1 + \mathsf{cl}(\alpha) + \mathsf{cl}(\beta)$

(c) $1 + \mathsf{cl}(\alpha) + \mathsf{cl}(\beta) = \mathsf{cl}(\alpha * \beta)$

Unbalanced brackets in a proper prefix of a formula

Problem: Every proper prefix of a well-formed formula has more opening brackets than closing brackets.

A proper prefix of φ is a non-empty segment of φ starting from the first symbol of φ and ending before the last symbol of φ.

How many proper prefixes does a formula have?

A proper prefix of φ is a non-empty segment of φ starting from the first symbol of φ and ending before the last symbol of φ.

1. Write down all the proper prefixes of $((\neg p) \wedge (q \rightarrow r))$.
2. Write down all the proper prefixes of $(\alpha \wedge \beta)$ where α and β are well-formed formulas and $*$ is a binary connective.