Introduction to Undecidability

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Lecture 23

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Outline

Learning Goals

By the end of this lecture, you should be able to:

Introduction to undecidability

- Define decision problem.
- Define decidable problem.
- Define undecidable problem.
- Prove that a decision problem is decidable by giving an algorithm to solve it.

The halting problem

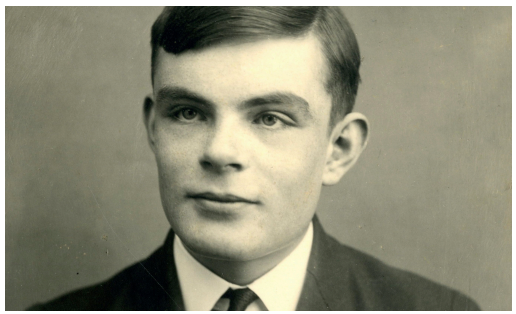
- Describe the halting problem.
- Prove that the halting problem is undecidable.

Exploring the limitation of computation

Most of CS focuses on what we CAN compute.

Are there problems that CANNOT be solved by a computer even with unlimited time and space?

The answer is yes. This was proved by Alan Turing in 1936.



What is a computer program/algorithm?

In the old days, there were no electronic computers. A computer refers to a person who computes. (Watch Hidden Figures.)

Turing's idea of a "computer program" was a list of instructions that a person could follow.

For us, an algorithm could refer to any of the following:

- Racket, C, and C++ programs
- Turing machines
- High-level pseudo-code

Decidable and undecidable problems

We focus on decision problems. **A decision problem** is a question with a yes/no answer.

An algorithm solves a problem iff it produces the correct output for the problem for every input.

A decision problem is

- **Decidable** iff there exists an algorithm that solves the problem.
- **Undecidable** iff there does not exist an algorithm that solves the problem.

CQ 1 Examples of decision problems

CQ 1: Given a Propositional formula, is it satisfiable?

(A) This problem is decidable.

(B) This problem is undecidable.

(C) I don't know.

CQ 2 Examples of decision problems

CQ 2: Given a Predicate formula, is it valid? (Take a guess. All answers will be marked correct.)

(A) This problem is decidable.

(B) This problem is undecidable.

(C) I don't know.

CQ 3 Examples of decision problems

CQ 3: Given a positive integer, is it prime?

(A) This problem is decidable.

(B) This problem is undecidable.

(C) I don't know.

CQ 4 Examples of decision problems

CQ 4: Given a Hoare triple, is it satisfied under partial correctness? (Take a guess. All answers will be marked correct.)

(A) This problem is decidable.

(B) This problem is undecidable.

(C) I don't know.

CQ 5 Examples of decision problems

CQ 5: Given a Hoare triple, is it satisfied under total correctness? (Take a guess. All answers will be marked correct.)

(A) This problem is decidable.

(B) This problem is undecidable.

(C) I don't know.

CQ 6 Examples of decision problems

CQ 6: Given a program P and an input I, does P terminate when run with the input I? (Take a guess. All answers will be marked correct.)

(A) This problem is decidable.

(B) This problem is undecidable.

(C) I don't know.

The Halting problem

The decision problem: Given a program P and an input I, will P halt when run with input I?

- "Halts" means "terminates" or " does not get stuck."
- One of the first known undecidable problems

The Halting problem is undecidable.

There does not exist an algorithm H, which gives the correct answer for the Halting problem for every program P and every input I.

Exercise: Translate the above statement into a Predicate formula.

The Halting Problem is Undecidable

A proof by video

https://www.youtube.com/watch?v=92WHN-pAFCs

Common questions about the video

- Why can we feed a program as an input to itself?
  We can convert any program to a string, then we can feed the string of the program to itself as input.
- What does the negator do?
  It negates the behaviour of the machine. If H predicts that the program halts, then the negator goes into an infinite loop and does not halt. If H predicts that the program does not halt, then the negator halts.
  The negator is designed to make H fail at its prediction task.
- Why do we need the photocopier?
  In the video, H takes two inputs. We need to make two copies of the input. In code, we do not need the photocopier. We simply need to call H(P,P).

The Halting Problem is Undecidable

Theorem: The Halting problem is undecidable.

Proof by contradiction.

Assume that there exists an algorithm H, which solves the Halting problem for every program and every input.

We will construct an algorithm X, which takes program P as input.

We will show that H gives the wrong answer when predicting whether the program X halts when run with input X. This contradicts the fact that H solves the Halting problem for every program and every input. Therefore, H does not exist.

□

Revisiting the learning goals

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