Program Verification
While Loops

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Lecture 20

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek
Outline

Program Verification: While Loops
  Learning Goals
  Proving Partial Correctness - Example 1
  Proving Partial Correctness - Example 2
  Proving Termination
  Revisiting the Learning Goals
Learning Goals

By the end of this lecture, you should be able to:

Partial correctness for while loops
- Determine whether a given formula is an invariant for a while loop.
- Find an invariant for a given while loop.
- Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops
- Determine whether a given formula is a variant for a while loop.
- Find a variant for a given while loop.
- Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.
Proving Total Correctness of While Loops

- Partial correctness
- Termination
Proving Partial Correctness of While Loops

\[
\begin{align*}
\langle P \rangle \\
\langle I \rangle \\
\text{while ( } & \text{B } \text{) } \\
\langle (I \land B) \rangle & \text{ partial - while } \\
C & \\
\langle I \rangle & <\text{justify based on C } - \text{ a subproof}> \\
\langle (I \land \neg B) \rangle & \text{ partial - while } \\
\langle Q \rangle & \text{ implied (B)}
\end{align*}
\]

Proof of implied (A): \( P \rightarrow I \)

Proof of implied (B): \((I \land (\neg B)) \rightarrow Q\)

I is called a loop invariant. We need to determine I!
What is a loop invariant?

A loop invariant is:

- A relationship among the variables. (A predicate formula involving the variables.)
- The word “invariant” means something that does not change.
- It is true before the loop begins.
- It is true at the start of every iteration of the loop and at the end of every iteration of the loop.
- It is true after the loop ends.
Proving partial correctness of while loops

Indicate the places in the program where the loop invariant is true.

\(\{ (x \geq 0) \}\)
y = 1;
z = 0;
            \textbf{while } (z \neq x) \{ \\
            \hspace{1em} z = z + 1; \hspace{1em}  \\
            \hspace{1em} y = y \times z; \\
\} \hspace{1em}  \\
\{ (y = x!) \}\)
Proving partial correctness of a while loop

Steps to follow:

- Find a loop invariant.
- Complete the annotations.
- Prove any implied’s.

How do we find a loop invariant???
How do we find a loop invariant?

First, we need to understand the purpose of an invariant.

- The postcondition is the ultimate goal of our while loop.
- At every iteration, we are making progress towards the postcondition.
- The invariant is describing the progress we are making at every iteration.
Example 1:
Prove that the following triple is satisfied under partial correctness.
\[
\{ (x \geq 0) \}\]
y = 1;
z = 0;
while (z \neq x) {
    z = z + 1;
    y = y \times z;
}
\{ (y = x!) \}\]
Finding a loop invariant

Step 1: Write down the values of all the variables every time the while test is reached.

\[\| (x \geq 0) \|\]

\[y = 1;\]
\[z = 0;\]
\[\textbf{while} \ (z \neq x) \ {\}
  \[z = z + 1;\]
  \[y = y * z;\]
\} \]
\[\| (y = x!) \|\]
Finding a loop invariant

Step 2: Find relationships among the variables that are true for every while test. These are our candidate invariants.

Come up with some invariants in the next 2 minutes.

\[
\begin{align*}
\mathbf{\Gamma} (x \geq 0) \\
y &= 1; \\
z &= 0; \\
\textbf{while (z != x) } \{ \\
& \quad z = z + 1; \\
& \quad y = y \times z; \\
\} \\
\mathbf{\Gamma} (y = x!)
\end{align*}
\]

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<td>120 = 5!</td>
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CQ 1: Is $\neg(z = x)$ a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[\{(x \geq 0)\}\]
\[y = 1;\]
\[z = 0;\]
\[\textbf{while } (z \neq x) \{\]
\[\quad z = z + 1;\]
\[\quad y = y * z;\]
\[\}\]
\[\{(y = x!)\}\]

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<td>$120 = 5!$</td>
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</table>
CQ 2: Is (z ≤ x) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[\begin{array}{l}
((x \geq 0)) \\
y = 1; \\
z = 0; \\
\textbf{while} (z != x) \{ \\
\quad z = z + 1; \\
\quad y = y \ast z; \\
\}\] 

\[\begin{array}{lll}
| x & z & y \!
\| \\
|---|---|---| \\
| 5 & 0 & 1 = 0! \\
| 5 & 1 & 1 = 1! \\
| 5 & 2 & 2 = 2! \\
| 5 & 3 & 6 = 3! \\
| 5 & 4 & 24 = 4! \\
| 5 & 5 & 120 = 5! \\
\end{array}\]
CQ 3: Is \( y = z! \) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\begin{align*}
\langle (x \geq 0) \rangle \\
y &= 1; \\
z &= 0; \\
\textbf{while} \ (z \neq x) \ {\{ \ }
    \quad z &= z + 1; \\
    \quad y &= y \ast z; \\
\} \\
\langle (y = x!) \rangle
\end{align*}
\]

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CQ 4 Is this a loop invariant?

CQ 4: Is \((y = x!)\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\begin{align*}
\ll l(x \geq 0) & \rr
\qquad y = 1; \\
\qquad z = 0; \\
\textbf{while} \quad (z \neq x) \quad \{ \\
\qquad \quad z = z + 1; \\
\qquad \quad y = y \ast z; \\
\quad \}\r
\end{align*}
\]

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</table>
CQ 5: Is \((z \leq x) \land (y = z!)\) a loop invariant?

(A) Yes (B) No (C) I don’t know...

\[
\{ (x \geq 0) \}
\]

\[
y = 1;
\]
\[
z = 0;
\]
\[
\textbf{while} \ (z \neq x) \ \{ \\
\hspace{1cm} z = z + 1; \\
\hspace{1cm} y = y \ast z;
\}
\]

\[
\} \ (y = x!)
\]

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Finding a loop invariant

Step 3: Try each candidate invariant until we find one that works for our proof.

\( ((x \geq 0)) \)

\[ y = 1; \]
\[ z = 0; \]
\[ \textbf{while} \ (z \neq x) \ { \}
\[ \quad z = z + 1; \]
\[ \quad y = y \times z; \]
\[ { \}} \]
\( ((y = x!)) \)

\[
\begin{array}{|c|c|c|}
\hline
x & z & y \\
\hline
5 & 0 & 1 = 0! \\
5 & 1 & 1 = 1! \\
5 & 2 & 2 = 2! \\
5 & 3 & 6 = 3! \\
5 & 4 & 24 = 4! \\
5 & 5 & 120 = 5! \\
\hline
\end{array}
\]
How do we find an invariant?

A recap of the steps to find an invariant:

- Write down the values of all the variables every time the while test is reached.
- Find relationships among the variables that are true for every while test. These are our candidate invariants.
- Try each candidate invariant until we find one that works for our proof.
Partial While - Example 1 \(((z \leq x)\text{ as the invariant})\)


given \(\vdash (x \geq 0)\) 
given \(\vdash (0 \leq x)\) implied (A)
y = 1; assignment
\(\vdash (0 \leq x)\) assignment
z = 0; assignment
\(\vdash (z \leq x)\)
while \((z \neq x)\) {
\(\vdash ((z \leq x) \land (\neg(z = x)))\) partial while
\(\vdash (z + 1 \leq x)\) implied (B)
z = z + 1;
\(\vdash (z \leq x)\) assignment
y = y \* z;
\(\vdash (z \leq x)\) assignment
}
\(\vdash ((z \leq x) \land (\neg(\neg(z = x)))\) partial while
\(\vdash (y = x!)\) implied (C)
We used \((z \leq x)\) as the invariant.

**CQ 7:** Is there a proof for implied \((A)\)?

\[ ((x \geq 0) \rightarrow (0 \leq x)) \]

\( (A) \) Yes  
\( (B) \) No  
\( (C) \) I don’t know.
CQ 8 Is there a proof for implied (B)?

We used \((z \leq x)\) as the invariant.

CQ 8: Is there a proof for implied (B)?

\[\(((z \leq x) \land \neg(z = x))) \rightarrow (z + 1 \leq x)))\]

(A) Yes
(B) No
(C) I don’t know.
CQ 9: Is there a proof for implied (C)?

We used \((z \leq x)\) as the invariant.

CQ 9: Is there a proof for implied (C)?

\[
(((z \leq x) \land \neg(\neg(z = x)))) \rightarrow (y = x!))
\]

(A) Yes
(B) No
(C) I don’t know.
Example 2:
Prove that the following triple is satisfied under partial correctness.

$\{ (x \geq 0) \} \quad y = 1; \quad z = 0; \quad \textbf{while} \ (z < x) \ \{ \quad \begin{array}{l} z = z + 1; \\ y = y \ast z; \end{array} \ \} \quad \{ (y = x!) \}$

Let’s try using ($y = z!$) as the invariant in our proof.
Which invariant leads to a valid proof?

To check whether an invariant leads to a valid proof, we need to check whether all of the implied’s can be proved.
CQ 11: Is there a proof for implied (A)?

We used \((y = z!\)) as the invariant.

CQ 11: Is there a proof for implied (A)?

\[ ((x \geq 0) \rightarrow (1 = 0!)) \]

(A) Yes
(B) No
(C) I don’t know.
We used \((y = z!\) as the invariant.

**CQ 12:** Is there a proof for implied (B)?

\[
(((y = z!) \land (z < x)) \rightarrow (y \ast (z + 1) = (z + 1)!))
\]

(A) Yes
(B) No
(C) I don’t know.
CQ 13 Is there a proof for implied (C)?

We used \((y = z!\) as the invariant.

CQ 13: Is there a proof for implied (C)?

\[((((y = z!) \land (\neg(z < x))) \rightarrow (y = x!))\]

(A) Yes

(B) No

(C) I don’t know.
We used \(((y = z!) \land (z \leq x))\) as the invariant.

**CQ 14:** Is there a proof for implied (C)?

\[
(((y = z!) \land (z \leq x)) \land (\neg(z < x))) \rightarrow (y = x!))
\]

(A) Yes
(B) No
(C) I don’t know.
Proving Termination

Find an integer expression that

- is non-negative before the loop starts, at every iteration of the loop, and after the loop ends.
- decreases by at least 1 at every iteration of the loop.

This integer expression is called a variant (something that changes).

The loop must terminate because a non-negative integer can decrease by 1 a finite number of times.
Example 2: Finding a variant

Example 2:
Prove that the following program terminates.

\[ y = 1; \]
\[ z = 0; \]
\[ \textbf{while} \ (z < x) \ { \}
\[ \hspace{1em} z = z + 1; \]
\[ \hspace{1em} y = y \times z; \]
\[ } \]

How do we find a variant? The loop guard \((z < x)\) helps.
Example 2: Proof of Termination

Consider the variant \((x - z)\).

Before the loop starts, \((x - z) \geq 0\) because the precondition is \((x \geq 0)\) and the second assignment mutates \(z\) to be 0.

During every iteration of the loop, \((x - z)\) decreases by 1 because \(x\) does not change and \(z\) increases by 1.

Thus, \(x - z\) will eventually reach 0.

When \(x - z = 0\), the loop guard \(z < x\) will terminate the loop.
Revisiting the learning goals

By the end of this lecture, you should be able to:
Partial correctness for while loops
  ▶ Determine whether a given formula is an invariant for a while loop.
  ▶ Find an invariant for a given while loop.
  ▶ Prove that a Hoare triple is satisfied under partial correctness for a program containing while loops.

Total correctness for while loops
  ▶ Determine whether a given formula is a variant for a while loop.
  ▶ Find a variant for a given while loop.
  ▶ Prove that a Hoare triple is satisfied under total correctness for a program containing while loops.