Predicate Logic: Natural Deduction

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Lecture 15

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek

Outline

Learning goals

By the end of this lecture, you should be able to:

- Describe the rules of inference for natural deduction.
- Prove that a conclusion follows from a set of premises using natural deduction inference rules.

CQ Forall-elimination

Suppose that our premise is $(\forall \mathrm{x}\ \alpha)$ where α is a well-formed predicate formula. Which of the following formulas can be conclude by applying $\forall$e on the premise?

(A) $\alpha[\mathrm{a}/\mathrm{x}]$

(B) $\alpha[\mathrm{y}/\mathrm{x}]$

(C) $\alpha[\mathrm{g}(\mathrm{b}, \mathrm{z})/\mathrm{x}]$

(D) Two of (A), (B), and (C)

(E) All of (A), (B), and (C)

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

CQ Exists-introduction

Proof 1:

1.	$(P(y) \rightarrow Q(y))$	premise
2.	$(\exists x\ (P(x) \rightarrow Q(y)))$	∃i: 1

Proof 2:

1.	$(P(y) \rightarrow Q(y))$	premise
2.	$(\exists x\ (P(x) \rightarrow Q(x)))$	∃i: 1

Which of the following is a correct application of the ∃i rule?

(A) Both proofs

(B) Proof 1 only

(C) Proof 2 only

(D) Neither proof

CQ Which rule should I apply first?

Suppose that we want to show that

$$\{(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash (\exists \mathrm{y}\ \mathrm{P}(\mathrm{y})).$$

Which rule would you apply first?

(A) I would apply $\forall$e on the premise first.

(B) I would apply $\exists$i to produce the conclusion first.

(C) Both (a) and (b) will eventually lead to valid solutions.

(D) I don't know...

CQ Forall-introduction

I want to prove that "every CS 245 student loves Natural Deduction."

Proof.
Pick an arbitrary CS 245 student. I happened to pick a student who loves chocolates. (Do some work....) Conclude that the student loves Natural Deduction. □

What can I conclude from the above proof?

(A) Every CS 245 student loves Natural Deduction.

(B) Every CS 245 student who loves chocolates, loves Natural Deduction.

(C) None of the above

CQ Which rule should I apply first?

Suppose that I want to show that

$$\{(\forall x\ (P(x) \wedge Q(x)))\} \vdash (\forall x\ (P(x) \rightarrow Q(x))).$$

As I am constructing the proof, which rule should I apply **first**? (Note that this may not be the rule that comes **first** in the completed proof.)

(A) $\forall$e on the premise

(B) $\forall$i to produce the conclusion

(C) Both will lead to valid solutions.

(D) Neither will lead to a valid solution.

CQ What's wrong with this proof?

Suppose that I want to show that

$$\{(\forall x(P(x) \land Q(x)))\} \vdash (\forall x(P(x) \rightarrow Q(x))).$$

Consider the following proof.

1.	$(\forall x(P(x) \land Q(x)))$	premise
2.	$(P(x_0) \land Q(x_0))$	$\forall$e: 1
3.	$Q(x_0)$	$\land$e: 2
4.	x_0 fresh	assumption
5.	$P(x_0)$	assumption
6.	$Q(x_0)$	reflexive: 3
7.	$(P(x_0) \rightarrow Q(x_0))$	$\rightarrow$i: 5-6
8.	$(\forall x(P(x) \rightarrow Q(x)))$	$\forall$i: 4-7

What's wrong with this proof?

CQ Which rule should I apply first?

Suppose that we want to show that

$$\{(\exists x\ ((\neg P(x)) \land (\neg Q(x))))\} \vdash (\exists x\ (\neg(P(x) \land Q(x)))).$$

As I am constructing the proof, which rule should I apply **first**? (Note that this may not be the rule that comes **first** in the completed proof.)

(A) $\exists$e on the premise

(B) $\exists$i to produce the conclusion

(C) Both (a) and (b) will lead to valid solutions.

(D) Neither will lead to a valid solution.

CQ Which rule should I apply first?

Suppose that I want to show that

$$\{(\forall \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \to \mathrm{Q}(\mathrm{x}))), (\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))\} \vdash (\exists \mathrm{x}\ \mathrm{Q}(\mathrm{x})).$$

As I am constructing the proof, which rule should I apply **first**? (Note that this may not be the rule that comes **first** in the completed proof.)

(A) $\forall$e

(B) $\exists$e

(C) $\forall$i

(D) $\exists$e

(E) I don't know.

CQ What's wrong with this proof?

Suppose that we want to show that

$$\{(\forall x\ (P(x) \to Q(x))), (\exists x\ P(x))\} \vdash (\exists x\ Q(x)).$$

Consider the following proof.

1.	$(\forall x\ (P(x) \to Q(x)))$	premise
2.	$(\exists x\ P(x))$	premise
3.	$(P(x_0) \to Q(x_0))$	$\forall$e: 1
4.	$P(x_0)$, x_0 fresh	assumption
5.	$Q(x_0)$	$\to$e: 3, 4
6.	$(\exists x\ Q(x))$	$\exists$i: 5
7.	$(\exists x\ Q(x))$	$\exists$e: 2, 4-6

What's wrong with this proof?

CQ Which rule should I apply first?

Suppose that I want to show that

$$\{(\exists x\ P(x)), (\forall x\ (\forall y\ (P(x) \rightarrow Q(y))))\} \vdash (\forall y\ Q(y)).$$

As I am constructing the proof, which rule should I apply **first**? (Note that this may not be the rule that comes **first** in the completed proof.)

(A) $\forall$e

(B) $\exists$e

(C) $\forall$i

(D) $\exists$e

(E) I don't know.

CQ Which rule should I apply first?

Suppose that we want to show that

$$\{(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x})), (\forall \mathrm{x}\ (\forall \mathrm{y}\ (\mathrm{P}(\mathrm{x}) \to \mathrm{Q}(\mathrm{y}))))\} \vdash (\forall \mathrm{y}\ \mathrm{Q}(\mathrm{y})).$$

As I am constructing the proof, which rule should I apply **second**? (Note that this may not be the rule that comes **second** in the completed proof.)

(A) $\forall$e

(B) $\exists$e

(C) $\forall$i

(D) $\exists$e

(E) I don't know.

CQ Which rule should I apply first?

Suppose that we want to show that

$$\{(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}, \mathrm{y})))\} \vdash (\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{P}(\mathrm{x}, \mathrm{y}))).$$

As I am constructing the proof, which rule should I apply **first**? (Note that this may not be the rule that comes **first** in the completed proof.)

(A) ∀e

(B) ∃e

(C) ∀i

(D) ∃e

(E) I don't know.

CQ Which rule should I apply first?

Suppose that we want to show that

$$\{(\exists y\ (\forall x\ P(x, y)))\} \vdash (\forall x\ (\exists y\ P(x, y))).$$

As I am constructing the proof, which rule should I apply **second**? (Note that this may not be the rule that comes **second** in the completed proof.)

(A) ∀e

(B) ∃e

(C) ∀i

(D) ∃e

(E) I don't know.

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Describe the rules of inference for natural deduction.
- Prove that a conclusion follows from a set of premises using natural deduction inference rules.