Predicate Logic: Semantics

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Lecture 13

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Outline

Learning goals

By the end of this lecture, you should be able to:

- Define interpretation.
- Define environment.
- Determine the truth value of a formula given an interpretation and an environment.
- Give an interpretation and an environment that make a formula true.
- Given an interpretation and an environment that make a formula false.
- Determine and justify whether a formula is valid, satisfiable, and/or unsatisfiable.

The Language of Predicate Logic

- Domain: a non-empty set of objects
- Constants: concrete objects in the domain
- Functions: takes objects in the domain as arguments and returns an object of the domain.
- Predicates: takes objects in the domain as arguments and returns true or false. They describe properties of objects or relationships between objects.
- Variables: placeholders for concrete objects in the domain
- Quantifiers: for how many objects in the domain is the statement true?

The semantics of a predicate formula

Given a well-formed formula of predicate logic, does the formula evaluate to F or T in some context?

Example: What does $(\mathrm{P}(\mathrm{a}) \vee \mathrm{Q}(\mathrm{a}, \mathrm{b}))$ mean?

The symbols $\mathrm{P}, \mathrm{Q}, \mathrm{a}$, and b do not have intrinsic meanings.

In **propositional logic**, a truth valuation is enough to assign a meaning to a formula.

In **predicate logic**, we need an interpretation, and possibly an environment.

Our language of predicate logic

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Terms without variables: a, $\mathrm{f}(\mathrm{a})$.

Formulas without variables: $\mathrm{P}(\mathrm{a})$, $\mathrm{Q}(\mathrm{a}, \mathrm{b})$, $(\neg \mathrm{P}(\mathrm{a}))$, $(\mathrm{P}(\mathrm{a}) \vee \mathrm{Q}(\mathrm{a}, \mathrm{b}))$.

Terms with variables: x, $\mathrm{f}(\mathrm{x})$.

Formulas with variables: $\mathrm{P}(\mathrm{x})$, $\mathrm{Q}(\mathrm{x}, \mathrm{b})$, $(\neg \mathrm{P}(\mathrm{x}))$, $(\mathrm{P}(\mathrm{x}) \vee \mathrm{Q}(\mathrm{x}, \mathrm{b}))$, $(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))$, $(\exists \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{b}))$, $(\forall \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))$, $(\exists \mathrm{x}\ (\mathrm{P}(\mathrm{x}) \vee \mathrm{Q}(\mathrm{x}, \mathrm{b})))$, $(\forall \mathrm{y}\ (\exists \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$.

Terms and formulas without variables

Consider a term or a formula that contains only constant, function, and predicate symbols.

Terms without variables:
a, $\mathrm{f(a)}$, $\mathrm{f(f(a))}$, $\mathrm{g(a, b)}$, $\mathrm{g(a, f(b))}$.

Formulas without variables:
$\mathrm{P(a)}$, $\mathrm{Q(a, b)}$, $(\neg \mathrm{P(a)})$, $(\mathrm{P(a)} \vee \mathrm{Q(a, b)})$, $\mathrm{P(g(a, b))}$,
$\mathrm{Q(a, f(f(a)))}$, $(\neg \mathrm{P(f(b))})$, $(\mathrm{P(f(b))} \vee \mathrm{Q(g(a, b), f(c))})$.

The semantics of terms/formulas without variables

Interpreting terms/formulas without variables requires an interpretation, which contains the following components:

- Domain
- A meaning for each constant symbol
- A meaning for each function symbol
- A meaning for each predicate symbol

An example of an interpretation

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Interpretation I:

- Domain: $\mathrm{dom}(\mathrm{I})$ is the set of integers.
- Constants: $\mathrm{a}^{\mathrm{I}} = 1$, $\mathrm{b}^{\mathrm{I}} = 2$, $\mathrm{c}^{\mathrm{I}} = 3$.
- Functions:
  f^{I}: $\mathrm{f}^{\mathrm{I}}(\mathrm{x}) = \mathrm{x} + 1$.
  g^{I}: $\mathrm{g}^{\mathrm{I}}(\mathrm{x}, \mathrm{y}) = \mathrm{x} + \mathrm{y}$.
- Predicates:
  P^{I}: $\mathrm{P}^{\mathrm{I}}(\mathrm{x})$ is true if and only if $\mathrm{x} > 5$.
  Q^{I}: $\mathrm{Q}^{\mathrm{I}}(\mathrm{x}, \mathrm{y})$ is true if and only if $\mathrm{x} > \mathrm{y}$.

Evaluating terms and formulas without variables

Evaluate these terms and formulas under I.
$\mathrm{f(f(a))}$, $\mathrm{g(a, f(b))}$, $\mathrm{Q(a, f(f(a)))}$, $\mathrm{(P(a) \vee Q(a, b))}$.

Interpretation I:

- Domain: $\mathrm{dom(I)}$ is the set of integers.
- Constants: $\mathrm{a^I} = 1$, $\mathrm{b^I} = 2$, $\mathrm{c^I} = 3$.
- Functions:
  $\mathrm{f^I}$: $\mathrm{f^I(x) = x + 1}$.
  $\mathrm{g^I}$: $\mathrm{g^I(x, y) = x + y}$.
- Predicates:
  $\mathrm{P^I}$: $\mathrm{P^I(x)}$ is true if and only if $\mathrm{x > 5}$.
  $\mathrm{Q^I}$: $\mathrm{Q^I(x, y)}$ is true if and only if $\mathrm{x > y}$.

A function must be total

A function symbol $f^{(k)}$ must be interpreted as a function f^I that is total on the domain D.

$$f^{(k)} : D \times D \to D$$

- Any k-tuple $(d_1, ..., d_k)$ in D^k can be an input to $f^{(k)}$.
- For any legal k-tuple, $f^I(d_1, ..., d_k)$ must be in D.

CQ A function must be total

Which of the following functions is total?

(A) $g^I(x, y) = x - y$. The domain is the set of natural numbers.
(B) $f^I(x) = \sqrt{x}$. The domain is the set of integers.
(C) $f^I(x) = x + 1$. The domain is $\{1, 2, 3\}$.
(D) $f^I(1) = 2$, $f^I(2) = 3$ and $f^I(3) = 3$. The domain is $\{1, 2, 3\}$.
(E) $g^I(x, y) = x > y$. The domain is the set of integers.

Notation for functions and predicates

Functions:

- g^I is the addition function. $g^I(x, y) = x + y$.
- $f^I(1) = 2$, $f^I(2) = 3$ and $f^I(3) = 3$.

Predicates:

- Q^I: $Q^I(x, y)$ is true if and only if $x > y$.
- $Q^I = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}$

Give an interpretation that makes the formula true/false

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Complete I such that

(A) $\mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{a})))^{\mathrm{I}} = \mathsf{T}$ or $\mathrm{I} \models \mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{a})))$

(B) $\mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{a})))^{\mathrm{I}} = \mathsf{F}$ or $\mathrm{I} \not\models \mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{a})))$

Interpretation I:

- Domain: $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- Constants: $\mathrm{a}^{\mathrm{I}} = ?$, $\mathrm{b}^{\mathrm{I}} = ?$, $\mathrm{c}^{\mathrm{I}} = ?$.
- Functions: $\mathrm{f}^{\mathrm{I}} : ?$, $\mathrm{g}^{\mathrm{I}} : ?$
- Predicates: $\mathrm{P}^{\mathrm{I}} : ?$, $\mathrm{Q}^{\mathrm{I}} : ?$

Terms and formulas w/o bound variables

Terms w/o bound variables:
$\mathrm{f}(\mathrm{f}(\mathrm{x}))$, $\mathrm{g}(\mathrm{x}, \mathrm{f}(\mathrm{b}))$.

Formulas w/o bound variables:
$\mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{x})))$, $(\mathrm{P}(\mathrm{f}(\mathrm{y})) \vee \mathrm{Q}(\mathrm{g}(\mathrm{x}, \mathrm{y}), \mathrm{f}(\mathrm{a})))$.

Interpreting terms/formulas with free variables requires an environment.

Environment

An environment is a function, which maps every variable symbol to a domain element.

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Let $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

An example of an environment for our language:
$\mathrm{E}(\mathrm{x}) = 1, \mathrm{E}(\mathrm{y}) = 2, \mathrm{E}(\mathrm{z}) = 2$.

Evaluating terms and formulas w/o bound variables

Evaluate these terms and formulas under I and E.
$\mathrm{f(f(x))}$, $\mathrm{g(x, f(b))}$, $\mathrm{Q(a, f(f(x)))}$, $\mathrm{(P(f(y)) \vee Q(g(x, y), f(a)))}$.

Environment E: $\mathrm{E(x) = 3, E(y) = 2, E(z) = 1}$.
Interpretation I:

- Domain: $\mathrm{dom(I)}$ is the set of integers.
- Constants: $\mathrm{a^I = 1}$, $\mathrm{b^I = 2}$, $\mathrm{c^I = 3}$.
- Functions:
  $\mathrm{f^I}$: $\mathrm{f^I(x) = x + 1}$.
  $\mathrm{g^I}$: $\mathrm{g^I(x, y) = x + y}$.
- Predicates:
  $\mathrm{P^I}$: $\mathrm{P^I(x)}$ is true if and only if $\mathrm{x > 5}$.
  $\mathrm{Q^I}$: $\mathrm{Q^I(x, y)}$ is true if and only if $\mathrm{x > y}$.

Give (I, E) that makes the formula true/false

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Complete I and E such that

(A) $\mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{x})))^{(\mathbf{I}, \mathbf{E})} = \mathsf{T}$ or $\mathrm{I} \models_{\mathrm{E}} \mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{x})))$

(B) $\mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{x})))^{(\mathbf{I}, \mathbf{E})} = \mathsf{F}$ or $\mathrm{I} \not\models_{\mathrm{E}} \mathrm{Q}(\mathrm{a}, \mathrm{f}(\mathrm{f}(\mathrm{x})))$

Interpretation I:

- Domain: $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- Constants: $\mathrm{a}^{\mathbf{I}} = ?$, $\mathrm{b}^{\mathbf{I}} = ?$, $\mathrm{c}^{\mathbf{I}} = ?$.
- Functions: $\mathrm{f}^{\mathbf{I}} : ?$, $\mathrm{g}^{\mathbf{I}} : ?$
- Predicates: $\mathrm{P}^{\mathbf{I}} : ?$, $\mathrm{Q}^{\mathbf{I}} : ?$

Important notes about an environment

- An environment has to map every variable symbol to a domain element, even if the variable does not appear in a formula.
- Bound variables get their meanings through the quantifiers. Free variables get their meanings through an environment.

  If a formula does not have any free variable, an interpretation is sufficient to give the formula meaning. There is no need to define an environment.

Formulas with quantifiers

The meanings of quantified formulas:

- $(\forall x\ \alpha)$: α is true for every possible value of x in the domain.

  $(\forall x\ \alpha)^{(I,E)} =$ T if $\alpha^{(I,E[x \mapsto d]))} =$ T for every $d \in \mathrm{dom}(I)$.
- $(\exists x\ \alpha)$: α is true for at least one value of x in the domain.

  $(\exists x\ \alpha)^{(I,E)} =$ T if $\alpha^{(I,E[x \mapsto d]))} =$ T for at least one $d \in \mathrm{dom}(I)$.

$E[x \mapsto d]$ is making a small change to the environment E.

The environment override/reassignment notation

$\mathrm{E}[\mathrm{x} \mapsto \mathrm{d}]$ keeps all the mappings in E intact
EXCEPT reassigning x to $\mathrm{d} \in \mathrm{dom}(\mathrm{I})$.

Suppose that $\mathrm{E}(\mathrm{x}) = 3$, $\mathrm{E}(\mathrm{y}) = 3$, $\mathrm{E}(\mathrm{z}) = 1$. $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.

1. $\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{x}) = ?$
2. $\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{y}) = ?$
3. $\mathrm{E}[\mathrm{x} \mapsto 2](\mathrm{z}) = ?$
4. $\mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{x}) = ?$
5. $\mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{y}) = ?$
6. $\mathrm{E}[\mathrm{x} \mapsto 2][\mathrm{y} \mapsto 1](\mathrm{z}) = ?$

Evaluate formulas under (I, E)

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Evaluate these terms and formulas under I and E.

(A) $(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$

(B) $(\exists \mathrm{x}\ (\forall \mathrm{y}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$

Interpretation I:

- Domain: $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- Predicates: $\mathrm{Q}^{\mathrm{I}} = \{\langle 1, 2\rangle, \langle 3, 1\rangle, \langle 2, 3\rangle\}$.

Give (I, E) that makes the formula true/false

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

Complete I and E such that the following formula is true/false.

(A) $(\forall \mathrm{y}\ (\exists \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$

(B) $(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{x}, \mathrm{y})))$

Interpretation I:

- Domain: $\mathrm{dom}(\mathrm{I}) = \{1, 2, 3\}$.
- ...

CQ Review Question 1

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

We want to show that the following predicate formula is satisfiable. What do we need to do?

$$\mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{b}, \mathrm{y}))$$

(A) Define an interpretation only.

(B) Define an environment only.

(C) Define an interpretation and an environment.

CQ Review Question 2

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

We want to show that the following predicate formula is satisfiable. What do we need to do?

$$(\forall \mathrm{x}\ \mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{b}, \mathrm{y})))$$

(A) Define an interpretation only.

(B) Define an environment only.

(C) Define an interpretation and an environment.

CQ Review Question 3

Our language of predicate logic: Constant symbols: $\mathrm{a}, \mathrm{b}, \mathrm{c}$. Variable symbols: $\mathrm{x}, \mathrm{y}, \mathrm{z}$. Function symbols: $\mathrm{f}^{(1)}, \mathrm{g}^{(2)}$. Predicate symbols: $\mathrm{P}^{(1)}, \mathrm{Q}^{(2)}$.

We want to show that the following predicate formula is satisfiable. What do we need to do?

$$(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{Q}(\mathrm{f}(\mathrm{x}), \mathrm{g}(\mathrm{b}, \mathrm{y}))))$$

(A) Define an interpretation only.

(B) Define an environment only.

(C) Define an interpretation and an environment.

Notation for Evaluating a Term

Consider the interpretation I below.

- $\mathrm{dom(I)} = \{1, 2, 3\}$.
- $\mathrm{a^I} = 2, \mathrm{b^I} = 1, \mathrm{c^I} = 1$.
- $\mathrm{f^I(x)} = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom(I)}$, $\mathrm{g^I(x)} = 1, \forall \mathrm{x} \in \mathrm{dom(I)}$.
- $\mathrm{Q^I} = \{\langle 1, 2\rangle\}$, $\mathrm{P^I} = \emptyset$.

Let the environment E be $\mathrm{E(x)} = 1$, $\mathrm{E(y)} = 2$, $\mathrm{E(z)} = 3$.

What is $\mathrm{g(a, x)^I}$?

$$\mathrm{g(a, x)^{(I,E)}} = \mathrm{g^I(a^I, E(x))} = \mathrm{g^I}(2, 1) = 1 \tag{1}$$

Notation for Evaluating a Formula

Consider the interpretation I below.

- $\mathrm{dom(I)} = \{1, 2, 3\}$.
- $\mathrm{a^I} = 2, \mathrm{b^I} = 1, \mathrm{c^I} = 1$.
- $\mathrm{f^I(x)} = \mathrm{x}, \forall \mathrm{x} \in \mathrm{dom(I)}$, $\mathrm{g^I(x)} = 1, \forall \mathrm{x} \in \mathrm{dom(I)}$.
- $\mathrm{Q^I} = \{\langle 1, 2 \rangle\}$, $\mathrm{P^I} = \emptyset$.

Let the environment E be $\mathrm{E(x)} = 1$, $\mathrm{E(y)} = 2$, $\mathrm{E(z)} = 3$.

$\mathrm{Q(x,y)}^{(\mathrm{I}, \mathrm{E}[\mathrm{y} \mapsto 2])} = \mathsf{T}$ because all of the following hold:

$$\mathrm{E}[\mathrm{y} \mapsto 2](\mathrm{x}) = 1 \quad (2)$$

$$\mathrm{E}[\mathrm{y} \mapsto 2](\mathrm{y}) = 2 \quad (3)$$

$$\langle \mathrm{E}[\mathrm{y} \mapsto 2](\mathrm{x}), \mathrm{E}[\mathrm{y} \mapsto 2](\mathrm{y}) \rangle = \langle 1, 2 \rangle \in \mathrm{Q^I} \quad (4)$$

Valid, Satisfiable, Unsatisfiable

A formula α is a valid:

$\mathrm{I} \models_{\mathrm{E}} \alpha$ for every interpretation I and environment E.

A formula α is unsatisfiable:

$\mathrm{I} \not\models_{\mathrm{E}} \alpha$ for every interpretation I and environment E.

A formula α is satisfiable:

$\mathrm{I} \models_{\mathrm{E}} \alpha$ for some interpretation I and environment E.

Most predicate formulas are satisfiable but not valid because we have a great deal of freedom to choose I and E.

Proving that a formula is valid/not valid

Consider the predicate formula $((\forall x\ P(x)) \rightarrow (\exists x\ P(x)))$.

- Determine whether the formula is valid or not.

- How do I prove that this formula is NOT valid?

- How do I prove that this formula is valid?

Revisiting the learning goals

By the end of this lecture, you should be able to:

- Define interpretation.
- Define environment.
- Determine the truth value of a formula given an interpretation and an environment.
- Give an interpretation and an environment that make a formula true.
- Given an interpretation and an environment that make a formula false.
- Determine and justify whether a formula is valid, satisfiable, and/or unsatisfiable.