Predicate Logic: Introduction and Translations

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Lecture 10

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Outline

Learning goals

By the end of this lecture, you should be able to

(Introduction to Predicate Logic)

- Give examples of English sentences that can be modeled using predicate logic but cannot be modeled using propositional logic.

(Translations)

- Translate an English sentence into a predicate formula.
- Translate a predicate formula into an English sentence.

What can't we express using propositional logic?

Can we express the following ideas using propositional logic?

- Translate this sentence: Alice is married to Jay and Alice is not married to Leon.
- Translate this sentence: Every bear likes honey.
- Define what it means for a natural number to be prime.

What can't we express using propositional logic?

A few things that are difficult to express using propositional logic:

- Relationships among individuals: Alice is married to Jay and Alice is not married to Leon.
- Generalizing patterns: Every bear likes honey.
- Infinite domains: Define what it means for a natural number to be prime.

We can use predicate logic (first-order logic) to express all of these.

Would you really use predicate logic?

Examples of predicate logic in CS245 so far:

1. Every well-formed formula has an equal number of left and right brackets.
2. For every truth valuation t, if all the premises are true under t, then the conclusion is true under t.
3. If there does not exist a natural deduction proof from the premises to the conclusion, then the premises do not semantically entail the conclusion.

Would you really use predicate logic?

Examples of predicate logic in Computer Science:

1. Data Structure: Every key stored in the left subtree of a node N is smaller than the key stored at N.
2. Algorithms: in the worst case, every comparison sort requires at least $\mathrm{cn} \log \mathrm{n}$ comparisons to sort n values, for some constant $\mathrm{c} > 0$.
3. Java example: there is no path via references from any variable in scope to any meomry location available for garbage collection...
4. Database query: select a person whose age is greater than or equal to the age of every other person in the table.

Elements of predicate logic

Predicate logic generalizes propositional logic.
New things in predicate logic:

- Domains
- Predicates
- Quantifiers

Domains

A domain is a non-empty set of objects. It is a world that our statement is situated within.
Examples of domains: natural numbers, people, animals, etc.
Why is it important to specify a domain? The same statement can have different truth values in different domains.
Consider this statement: There exists a number whose square is 2.

- If our domain is the set of natural numbers, is this statement true or false?
- If our domain is the set of real numbers, is this statement true or false?

Objects in a domain

Constants: concrete objects in the domain

- Natural numbers: 0, 6, 100, ...
- Alice, Bob, Eve, ...
- Animals: Winnie the Pooh, Micky Mouse, Simba, ...

Variables: placeholders for concrete objects, e.g. x, y, z.
A variable lets us refer to an object without specifying which particular object it is.

Predicates

A predicate represents

- a property of an individual, or
- a relationship among mulitple individuals.

Think of a predicate as a special type of function which takes constants and/or variables as inputs and outputs T/F. It can have any number of arguments.
Examples:

- Define $\mathrm{L(x)}$ to mean "x is a lecturer". (unary predicate)
  - Alice is a lecturer: $\mathrm{L(Alice)}$
  - Micky Mouse is not a lecturer: $(\neg \mathrm{L(MickyMouse)})$
  - y is a lecturer: $\mathrm{L(y)}$
- Define $\mathrm{Y(x, y)}$ to mean "x is younger than y". (binary predicate/relation)
  - Alex is younger than Sam: $\mathrm{Y(Alex, Sam)}$
  - a is younger than b: $\mathrm{Y(a, b)}$

Quantifiers

For how many objects in the domain is the statement true?

- The universal quantifier $\forall$: the statement is true for every object in the domain.
- The existential quantifier $\exists$: the statement is true for one or more objects in the domain.

CQ Translating English into Predicate Logic

Let the domain be the set of animals. $\mathrm{H(x)}$ means that x likes honey. $\mathrm{B(x)}$ means that x is a bear.
Consider the following translations of English sentences into predicate logic formulas. Are the translations correct?

1. At least one animal likes honey. $(\exists \mathrm{x}\ \mathrm{H(x)})$.
2. Not every animal likes honey. $(\neg(\forall \mathrm{x}\ \mathrm{H(x)}))$.

(A) Both are correct.
(B) 1 is correct and 2 is wrong.
(C) 1 is wrong and 2 is correct.
(D) Both are wrong.

Translating English into Predicate Logic

Translate the following sentences into predicate logic.

1. All animals like honey.
2. At least one animal likes honey.
3. Not every animal likes honey.
4. No animal likes honey.

Let the domain be the set of animals. $\mathrm{H(x)}$ means that x likes honey. $\mathrm{B(x)}$ means that x is a bear.

Translating English into Predicate Logic

Translate the following sentences into predicate logic.

5. No animal dislikes honey.
6. Not every animal dislikes honey.
7. Some animal dislikes honey.
8. Every animal dislikes honey.

Let the domain be the set of animals. $\mathrm{H(x)}$ means that x likes honey. $\mathrm{B(x)}$ means that x is a bear.

CQ Translating English into Predicate Logic

Consider this sentence: every bear likes honey. Which one is the correct translation into predicate logic?

(A) $(\forall x\ (B(x) \wedge H(x)))$

(B) $(\forall x\ (B(x) \vee H(x)))$

(C) $(\forall x\ (B(x) \rightarrow H(x)))$

(D) $(\forall x\ (H(x) \rightarrow B(x)))$

Let the domain be the set of animals. $H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

CQ Translating English into Predicate Logic

Consider this sentence: some bear likes honey. Which one is the correct translation into predicate logic?

(A) $(\exists x\ (B(x) \land H(x)))$

(B) $(\exists x\ (B(x) \lor H(x)))$

(C) $(\exists x\ (B(x) \to H(x)))$

(D) $(\exists x\ (H(x) \to B(x)))$

Let the domain be the set of animals. $H(x)$ means that x likes honey. $B(x)$ means that x is a bear.

Multiple Quantifiers

Let the domain be the set of people. Let $\mathrm{L}(\mathrm{x}, \mathrm{y})$ mean that person x likes person y.

Translate the following formulas into English.

1. $(\forall \mathrm{x}\ (\forall \mathrm{y}\ \mathrm{L}(\mathrm{x}, \mathrm{y})))$
2. $(\exists \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{L}(\mathrm{x}, \mathrm{y})))$
3. $(\forall \mathrm{x}\ (\exists \mathrm{y}\ \mathrm{L}(\mathrm{x}, \mathrm{y})))$
4. $(\exists \mathrm{y}\ (\forall \mathrm{x}\ \mathrm{L}(\mathrm{x}, \mathrm{y})))$

CQ Translations

Let the domain contain the set of all students and courses. Define the following predicates: $\mathrm{S(x)}$: x is a student. $\mathrm{C(y)}$: y is a course. $\mathrm{T(x, y)}$: student x has taken course y.

Translate the sentence "every student has taken some course." into a predicate formula. Let x refer to a student and let y refer to a course. What quantifiers should we use for x and y?

(A) $\forall \mathrm{x}$ and $\forall \mathrm{y}$

(B) $\forall \mathrm{x}$ and $\exists \mathrm{y}$

(C) $\exists \mathrm{x}$ and $\forall \mathrm{y}$

(D) $\exists \mathrm{x}$ and $\exists \mathrm{y}$

CQ Translations

Let the domain contain the set of all students and courses. Define the following predicates: $\mathrm{S}(\mathrm{x})$: x is a student. $\mathrm{C}(\mathrm{y})$: y is a course. $\mathrm{T}(\mathrm{x}, \mathrm{y})$: student x has taken course y.

Translate the sentence "every student has taken some course." into a predicate formula. Which of the following is a correct translation?

(A) $(\forall \mathrm{x}\ (\mathrm{S}(\mathrm{x}) \rightarrow (\exists \mathrm{y}\ (\mathrm{C}(\mathrm{y}) \rightarrow \mathrm{T}(\mathrm{x}, \mathrm{y})))))$

(B) $(\forall \mathrm{x}\ (\mathrm{S}(\mathrm{x}) \rightarrow (\exists \mathrm{y}\ (\mathrm{C}(\mathrm{y}) \wedge \mathrm{T}(\mathrm{x}, \mathrm{y})))))$

(C) $(\forall \mathrm{x}\ (\mathrm{S}(\mathrm{x}) \wedge (\exists \mathrm{y}\ (\mathrm{C}(\mathrm{y}) \rightarrow \mathrm{T}(\mathrm{x}, \mathrm{y})))))$

(D) $(\forall \mathrm{x}\ (\mathrm{S}(\mathrm{x}) \wedge (\exists \mathrm{y}\ (\mathrm{C}(\mathrm{y}) \wedge \mathrm{T}(\mathrm{x}, \mathrm{y})))))$

CQ Translations

Let the domain contain the set of all students and courses. Define the following predicates: $\mathrm{S}(\mathrm{x})$: x is a student. $\mathrm{C}(\mathrm{y})$: y is a course. $\mathrm{T}(\mathrm{x}, \mathrm{y})$: student x has taken course y.

Translate the sentence "some student has not taken any course." into a predicate formula. Let x refer to a student and let y refer to a course. What quantifiers should we use for x and y?

(A) $\forall \mathrm{x}$ and $\forall \mathrm{y}$

(B) $\forall \mathrm{x}$ and $\exists \mathrm{y}$

(C) $\exists \mathrm{x}$ and $\forall \mathrm{y}$

(D) $\exists \mathrm{x}$ and $\exists \mathrm{y}$

CQ Translations

Let the domain contain the set of all students and courses. Define the following predicates: $\mathrm{S(x)}$: x is a student. $\mathrm{C(y)}$: y is a course. $\mathrm{T(x,y)}$: student x has taken course y.

Translate the sentence "some student has not taken any course." into a predicate formula. Which of the following is a correct translation?

(A) $(\exists \mathrm{x}\ (\mathrm{S(x)} \rightarrow (\forall \mathrm{y}\ (\mathrm{C(y)} \rightarrow (\neg \mathrm{T(x,y)})))))$

(B) $(\exists \mathrm{x}\ (\mathrm{S(x)} \rightarrow (\forall \mathrm{y}\ (\mathrm{C(y)} \wedge (\neg \mathrm{T(x,y)})))))$

(C) $(\exists \mathrm{x}\ (\mathrm{S(x)} \wedge (\forall \mathrm{y}\ (\mathrm{C(y)} \rightarrow (\neg \mathrm{T(x,y)})))))$

(D) $(\exists \mathrm{x}\ (\mathrm{S(x)} \wedge (\forall \mathrm{y}\ (\mathrm{C(y)} \wedge (\neg \mathrm{T(x,y)})))))$

CQ Interpreting the quantifiers

Let the domain be $\{\mathrm{Alice}, \mathrm{Bob}, \mathrm{Eve}\}$. Let $\mathrm{C}(\mathrm{x})$ mean that person x likes chocolates.

Which of the following is equivalent to $(\forall \mathrm{x}\ \mathrm{C}(\mathrm{x}))$?

(A) $((\mathrm{C}(\mathrm{Alice}) \wedge \mathrm{C}(\mathrm{Bob})) \wedge \mathrm{C}(\mathrm{Eve}))$

(B) $((\mathrm{C}(\mathrm{Alice}) \vee \mathrm{C}(\mathrm{Bob})) \vee \mathrm{C}(\mathrm{Eve}))$

(C) $((\mathrm{C}(\mathrm{Alice}) \rightarrow \mathrm{C}(\mathrm{Bob})) \rightarrow \mathrm{C}(\mathrm{Eve}))$

(D) $((\mathrm{C}(\mathrm{Alice}) \leftrightarrow \mathrm{C}(\mathrm{Bob})) \leftrightarrow \mathrm{C}(\mathrm{Eve}))$

(E) None of the above

CQ Interpreting the quantifiers

Let the domain be $\{\mathrm{Alice}, \mathrm{Bob}, \mathrm{Eve}\}$. Let $\mathrm{C}(\mathrm{x})$ mean that person x likes chocolates.

Which of the following is equivalent to $(\exists \mathrm{x}\ \mathrm{C}(\mathrm{x}))$?

(A) $((\mathrm{C}(\mathrm{Alice}) \wedge \mathrm{C}(\mathrm{Bob})) \wedge \mathrm{C}(\mathrm{Eve}))$

(B) $((\mathrm{C}(\mathrm{Alice}) \vee \mathrm{C}(\mathrm{Bob})) \vee \mathrm{C}(\mathrm{Eve}))$

(C) $((\mathrm{C}(\mathrm{Alice}) \rightarrow \mathrm{C}(\mathrm{Bob})) \rightarrow \mathrm{C}(\mathrm{Eve}))$

(D) $((\mathrm{C}(\mathrm{Alice}) \leftrightarrow \mathrm{C}(\mathrm{Bob})) \leftrightarrow \mathrm{C}(\mathrm{Eve}))$

(E) None of the above

Interpreting the quantifiers

Let $\mathrm{D} = \{\mathrm{d_1}, \mathrm{d_2}, ..., \mathrm{d_n}\}$.

$\forall$ is a big AND:

$(\forall \mathrm{x}\ \mathrm{P(x)})$ is equivalent to $(...(\mathrm{P(d_1)} \wedge \mathrm{P(d_2)}) \wedge ... \wedge \mathrm{P(d_n)})$.

$\exists$ is a big OR:

$(\exists \mathrm{x}\ \mathrm{P(x)})$ is equivalent to $(...(\mathrm{P(d_1)} \vee \mathrm{P(d_2)}) \vee ... \vee \mathrm{P(d_n)})$.

Negating a quantifier formula

"Generalized De Morgan's Laws"

- $(\neg(\forall \mathrm{x}\ \mathrm{P}(\mathrm{x}))) \equiv (\exists \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))$
- $(\neg(\exists \mathrm{x}\ \mathrm{P}(\mathrm{x}))) \equiv (\forall \mathrm{x}\ (\neg \mathrm{P}(\mathrm{x})))$

At least, at most, and exactly

Let the domain be the set of animals. Let $\mathrm{B}(\mathrm{x})$ be that x is a bear.

1. There are at least two bears.
2. There are at most one bear.
3. There are exactly one bear.

Revisiting the learning goals

By the end of this lecture, you should be able to

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