Propositional Logic: Soundness and Completeness of Natural Deduction

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Lecture 7

Learning goals

Soundness and completeness of natural deduction

- Define soundness and completeness.
- Prove that a semantic entailment holds using natural deduction and the soundness of natural deduction.
- Show that no natural deduction proof exists using the contrapositive of the soundness of natural deduction.

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Soundness and Completeness of Natural Deduction

We want to prove that Natural Deduction is both sound and complete.

Soundness of Natural Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

$$\text{If } \Sigma \vdash \alpha \text{, then } \Sigma \models \alpha$$

Completeness of Natural Deduction means that all logical consequences in propositional logic are provable in Natural Deduction. That is,

$$\text{If } \Sigma \models \alpha \text{, then } \Sigma \vdash_{\mathrm{ND}} \alpha$$

Proof of Soundness

To prove soundness, we use induction on the length of the proof:

> For all deductions $\Sigma \vdash \alpha$ which have a proof of length n or less, it is the case that $\Sigma \models \alpha$.

That property, however, is not quite good enough to carry out the induction. We actually use the following property of a natural number n.

> Suppose that a formula α appears at line n of a partial deduction, which may have one or more open sub-proofs. Let Σ be the set of premises used and Γ be the set of assumptions of open sub-proofs. Then $\Sigma \cup \Gamma \models \alpha$.

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Basis of the Induction

Base case. The shortest deductions have length 1, and thus are either

1.	α	Premise.

or

1.	α	Assumption.
2.		

We have either $\alpha \in \Sigma$ (in the first case), or $\alpha \in \Gamma$ (in the second case).

Thus $\Sigma \cup \Gamma \vDash \alpha$, as required.

Proof of Soundness: Inductive Step

Inductive step. Hypothesis: the property holds for each $n < k$; that is,

> If some formula α appears at line k or earlier of some partial deduction, with premises Σ and un-closed assumptions Γ, then $\Sigma \cup \Gamma \models \alpha$.

To prove: if α' appears at line $k + 1$, then $\Sigma \cup \Gamma' \models \alpha'$
(where $\Gamma' = \Gamma \cup \alpha'$ when α' is an assumption, and $\Gamma' = \Gamma$ otherwise).

The case that α' is an assumption is trivial.

Otherwise, formula α' must have a justification by some rule. We shall consider each possible rule.

Inductive Step, Case I

Case I: α' was justified by $\wedge$i.

We must have $\alpha' = \alpha_1 \wedge \alpha_2$, where each of α_1 and α_2 appear earlier in the proof, at steps m_1 and m_2, respectively. Also, any sub-proof open at step m_1 or m_2 is still open at step $k+1$.

Thus the induction hypothesis applies to both; that is, $\Sigma \cup \Gamma \vDash \alpha_1$ and $\Sigma \cup \Gamma \vDash \alpha_2$.

By the definition of $\vDash$, this yields $\Sigma \cup \Gamma \vDash \alpha'$, as required.

Inductive Step, Case II

Case II: α' was justified by $\rightarrow$i.

Rule $\rightarrow$i requires that $\alpha' = \alpha_1 \rightarrow \alpha_2$ and there is a closed sub-proof with assumption α_1 and conclusion α_2, ending by step k. Also, any sub-proof open before the assumption of α_1 is still open at step $k + 1$.

The induction hypothesis thus implies $\Sigma \cup (\Gamma \cup \{\alpha_1\}) \vDash \alpha_2$.

Hence $\Sigma \cup \Gamma \vDash \alpha_1 \rightarrow \alpha_2$, as required.

Inductive Step, Cases III ff.

Case III: α' was justified by ¬e.

This requires that α' be the pseudo-formula $\perp$, and that the proof contain formulas α and $(\neg\alpha)$ for some α, each using at most k steps.

By the induction hypothesis, both $\Sigma \models \alpha$ and $\Sigma \models (\neg\alpha)$.

Thus Σ is contradictory, and $\Sigma \models \alpha'$ for any α'.

Cases IV–XIII:

The other cases follow by similar reasoning.

This completes the inductive step, and the proof of soundness.

Completeness of Natural Deduction

We now turn to completeness.

Recall that completeness means the following.

Let Σ be a set of formulas and φ be a formula.

$$\text{If } \Sigma \models \varphi, \text{ then } \Sigma \vdash \varphi.$$

That is, every consequence has a proof.

How can we prove this?

Proof of Completeness: Getting started

We shall assume that the set Σ of hypotheses is finite.
The theorem is also true for infinite sets of hypotheses, but that requires a completely different proof.

Suppose that $\Sigma \vDash \varphi$, where $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$.
Thus the formula $(\sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_m) \rightarrow \varphi$ is a tautology.

Lemma. Every tautology is provable in Natural Deduction.

Once we prove the Lemma, the result follows. Given a proof of $(\sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_m) \rightarrow \varphi$, one can use $\wedge$i and $\rightarrow$e to complete a proof of $\Sigma \vdash \varphi$.

Tautologies Have Proofs

For a tautology, every line of its truth table ends with T.
We can mimic the construction of a truth table using inferences in Natural Deduction.

> Claim. Let φ have k variables $p_1, \ldots, p_k$. Let v be a valuation, and define $\ell_1, \ell_2, \ldots, \ell_k$ as
>
> $$\ell_i = \begin{cases} p_i & \text{if } v(p_i) = \mathsf{T} \\ \neg p_i & \text{if } v(p_i) = \mathsf{F}. \end{cases}$$
>
> If $\varphi^v = \mathsf{T}$, then $\{\ell_1, \ldots \ell_k\} \vdash \varphi$, and
> if $\varphi^v = \mathsf{F}$, then $\{\ell_1, \ldots \ell_k\} \vdash (\neg\varphi)$.

To prove the claim, use structural induction on formulas
(which is induction on the column number of the truth table).

Once the claim is proven, we can prove a tautology as follows....

Outline of the Proof of a Tautology

1.	$p_1 \vee (\neg p_1)$	L.E.M.
2.	$p_2 \vee (\neg p_2)$	L.E.M.
$\vdots$	$\vdots$	
k.	$p_k \vee (\neg p_k)$	L.E.M.
k + 1.	p_1	assumption
	p_2	assumption
	$\vdots$	
	φ	
	$(\neg p_2)$	assumption
	$\vdots$	
	φ	
m.	φ	$\vee$e: 2, ...

m + 1.	$(\neg p_1)$	assumption
	$\vdots$	
	$\vdots$	
	φ	$\vee$e: m + 1, ...
n.	φ	$\vee$e: 1, m − (k + 1), n − (m + 1)

Once each variable is assumed true or false, the previous claim provides a proof.

Proving the Claim

Hypothesis: the following hold for formulas α and β:

If $\{\ell_1, \ldots, \ell_k\} \vDash \alpha$, then $\{\ell_1, \ldots, \ell_k\} \vdash \alpha$;
If $\{\ell_1, \ldots, \ell_k\} \nvDash \alpha$, then $\{\ell_1, \ldots, \ell_k\} \vdash (\neg\alpha)$;
If $\{\ell_1, \ldots, \ell_k\} \vDash \beta$, then $\{\ell_1, \ldots, \ell_k\} \vdash \beta$; and
If $\{\ell_1, \ldots, \ell_k\} \nvDash \beta$, then $\{\ell_1, \ldots, \ell_k\} \vdash (\neg\beta)$.

If $\{\ell_1, \ldots, \ell_k\} \vDash (\alpha \wedge \beta)$, put the two proofs of α and β together, and then infer $(\alpha \wedge \beta)$, by $\wedge$i.

If $\{\ell_1, \ldots, \ell_k\} \nvDash (\alpha \rightarrow \beta)$ (i.e., $\{\ell_1, \ldots, \ell_k\} \vDash \alpha$ and $\{\ell_1, \ldots, \ell_k\} \nvDash \beta$),

- Prove α and $(\neg\beta)$.
- Assume $(\alpha \rightarrow \beta)$; from it, conclude β ($\rightarrow$e) and then $\bot$ ($\neg$e).
- From the sub-proof, conclude $(\neg(\alpha \rightarrow \beta))$, by $\neg$i.

The other cases are similar.