# Propositional Logic: Soundness and Completeness of Natural Deduction

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Lecture 7

# Learning goals

Soundness and completeness of natural deduction

- Define soundness and completeness.
- Prove that a semantic entailment holds using natural deduction and the soundness of natural deduction.
- Show that no natural deduction proof exists using the contrapositive of the soundness of natural deduction.

# Soundness and Completeness of Natural Deduction

We want to prove that Natural Deduction is both sound and complete.

Soundness of Natural Deduction means that the conclusion of a proof is always a logical consequence of the premises. That is,

$$\text{If } \Sigma \vdash \alpha, \text{ then } \Sigma \models \alpha$$

Completeness of Natural Deduction means that all logical consequences in propositional logic are provable in Natural Deduction. That is,

$$\text{If } \Sigma \models \alpha, \text{ then } \Sigma \vdash_{ND} \alpha$$

# Proof of Soundness

To prove soundness, we use induction on the length of the proof:

> For all deductions $\Sigma \vdash \alpha$ which have a proof of length *n* or less, it is the case that $\Sigma \models \alpha$.

That property, however, is not quite good enough to carry out the induction. We actually use the following property of a natural number *n*.

> Suppose that a formula $\alpha$ appears at line *n* of a partial deduction, which may have one or more open sub-proofs. Let $\Sigma$ be the set of premises used and $\Gamma$ be the set of assumptions of open sub-proofs. Then $\Sigma \cup \Gamma \models \alpha$.


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# Basis of the Induction

**Base case.** The shortest deductions have length 1, and thus are either

| | | |
|---|---|---|
| 1. | $\alpha$ | Premise. |

or

| | | |
|---|---|---|
| 1. | $\alpha$ | Assumption. |
| 2. | | |

We have either $\alpha \in \Sigma$ (in the first case), or $\alpha \in \Gamma$ (in the second case).

Thus $\Sigma \cup \Gamma \models \alpha$, as required.

# Proof of Soundness: Inductive Step

**Inductive step.** Hypothesis: the property holds for each $n < k$; that is,

> If some formula $\alpha$ appears at line $k$ or earlier of some partial deduction, with premises $\Sigma$ and un-closed assumptions $\Gamma$, then $\Sigma \cup \Gamma \models \alpha$.

To prove: if $\alpha'$ appears at line $k + 1$, then $\Sigma \cup \Gamma' \models \alpha'$
(where $\Gamma' = \Gamma \cup \alpha'$ when $\alpha'$ is an assumption, and $\Gamma' = \Gamma$ otherwise).

The case that $\alpha'$ is an assumption is trivial.

Otherwise, formula $\alpha'$ must have a justification by some rule. We shall consider each possible rule.

# Inductive Step, Case I

**Case I:** $\alpha'$ was justified by $\wedge$i.

We must have $\alpha' = \alpha_1 \wedge \alpha_2$, where each of $\alpha_1$ and $\alpha_2$ appear earlier in the proof, at steps $m_1$ and $m_2$, respectively. Also, any sub-proof open at step $m_1$ or $m_2$ is still open at step $k + 1$.

Thus the induction hypothesis applies to both; that is, $\Sigma \cup \Gamma \models \alpha_1$ and $\Sigma \cup \Gamma \models \alpha_2$.

By the definition of $\models$, this yields $\Sigma \cup \Gamma \models \alpha'$, as required.


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# Inductive Step, Case II

**Case II**: $\alpha'$ was justified by $\rightarrow$i.

Rule $\rightarrow$i requires that $\alpha' = \alpha_1 \rightarrow \alpha_2$ and there is a closed sub-proof with assumption $\alpha_1$ and conclusion $\alpha_2$, ending by step $k$. Also, any sub-proof open before the assumption of $\alpha_1$ is still open at step $k+1$.

The induction hypothesis thus implies $\Sigma \cup (\Gamma \cup \{\alpha_1\}) \models \alpha_2$.

Hence $\Sigma \cup \Gamma \models \alpha_1 \rightarrow \alpha_2$, as required.

# Inductive Step, Cases III ff.

**Case III**: $\alpha'$ was justified by $\neg$e.

This requires that $\alpha'$ be the pseudo-formula $\perp$, and that the proof contain formulas $\alpha$ and $(\neg\alpha)$ for some $\alpha$, each using at most *k* steps.

By the induction hypothesis, both $\Sigma \models \alpha$ and $\Sigma \models (\neg\alpha)$.

Thus $\Sigma$ is contradictory, and $\Sigma \models \alpha'$ for any $\alpha'$.

**Cases IV–XIII:**

The other cases follow by similar reasoning.

This completes the inductive step, and the proof of soundness.

# Completeness of Natural Deduction

We now turn to completeness.

Recall that completeness means the following.

Let $\Sigma$ be a set of formulas and $\varphi$ be a formula.

If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

That is, every consequence has a proof.

How can we prove this?

# Proof of Completeness: Getting started

*We shall assume that the set $\Sigma$ of hypotheses is finite.*
*The theorem is also true for infinite sets of hypotheses, but that requires a completely different proof.*

Suppose that $\Sigma \models \varphi$, where $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$.
Thus the formula $(\sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_m) \rightarrow \varphi$ is a tautology.

Lemma. Every tautology is provable in Natural Deduction.

Once we prove the Lemma, the result follows. Given a proof of $(\sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_m) \rightarrow \varphi$, one can use $\wedge$i and $\rightarrow$e to complete a proof of $\Sigma \vdash \varphi$.

# Tautologies Have Proofs

For a tautology, every line of its truth table ends with T.
We can mimic the construction of a truth table using inferences in Natural Deduction.

Claim. Let $\varphi$ have $k$ variables $p_1, \ldots, p_k$. Let $v$ be a valuation, and define $\ell_1, \ell_2, \ldots, \ell_k$ as

$$\ell_i = \begin{cases} p_i & \text{if } v(p_i) = \mathrm{T} \\ \neg p_i & \text{if } v(p_i) = \mathrm{F}. \end{cases}$$

If $\varphi^v = \mathrm{T}$, then $\{\ell_1, \ldots \ell_k\} \vdash \varphi$, and
if $\varphi^v = \mathrm{F}$, then $\{\ell_1, \ldots \ell_k\} \vdash (\neg\varphi)$.

To prove the claim, use structural induction on formulas
(which is induction on the column number of the truth table).

Once the claim is proven, we can prove a tautology as follows....

# Outline of the Proof of a Tautology

| | | |
|---|---|---|
| 1. | $p_1 \vee (\neg p_1)$ | L.E.M. |
| 2. | $p_2 \vee (\neg p_2)$ | L.E.M. |
| $\vdots$ | $\vdots$ | |
| $k$. | $p_k \vee (\neg p_k)$ | L.E.M. |
| $k+1$. | $p_1$ | assumption |
| | $p_2$ | assumption |
| | $\vdots$ | |
| | $\varphi$ | |
| | $(\neg p_2)$ | assumption |
| | $\vdots$ | |
| | $\varphi$ | |
| $m$. | $\varphi$ | $\vee$e: 2, ... |

| | | |
|---|---|---|
| $m+1$. | $(\neg p_1)$ | assumption |
| | $\vdots$ | |
| | $\vdots$ | |
| | $\varphi$ | $\vee$e: $m+1$, ... |
| $n$. | $\varphi$ | $\vee$e: 1, $m-(k+1)$, $n-(m+1)$ |

Once each variable is assumed true or false, the previous claim provides a proof.

# Proving the Claim

Hypothesis: the following hold for formulas $\alpha$ and $\beta$:

If $\{\ell_1, \ldots, \ell_k\} \models \alpha$, then $\{\ell_1, \ldots, \ell_k\} \vdash \alpha$;
If $\{\ell_1, \ldots, \ell_k\} \not\models \alpha$, then $\{\ell_1, \ldots, \ell_k\} \vdash (\neg\alpha)$;
If $\{\ell_1, \ldots, \ell_k\} \models \beta$, then $\{\ell_1, \ldots, \ell_k\} \vdash \beta$; and
If $\{\ell_1, \ldots, \ell_k\} \not\models \beta$, then $\{\ell_1, \ldots, \ell_k\} \vdash (\neg\beta)$.

If $\{\ell_1, \ldots, \ell_k\} \models (\alpha \wedge \beta)$, put the two proofs of $\alpha$ and $\beta$ together, and then infer $(\alpha \wedge \beta)$, by $\wedge$i.

If $\{\ell_1, \ldots, \ell_k\} \not\models (\alpha \rightarrow \beta)$ (i.e., $\{\ell_1, \ldots, \ell_k\} \models \alpha$ and $\{\ell_1, \ldots, \ell_k\} \not\models \beta$),

- Prove $\alpha$ and $(\neg\beta)$.
- Assume $(\alpha \rightarrow \beta)$; from it, conclude $\beta$ ($\rightarrow$e) and then $\perp$ ($\neg$e).
- From the sub-proof, conclude $(\neg(\alpha \rightarrow \beta))$, by $\neg$i.

The other cases are similar.