Propositional Logic:
Semantic Entailment

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Lecture 6

Based on work by J. Buss, L. Kari, A. Lubiw, B. Bonakdarpour, D. Maftuleac, C. Roberts, R. Trefler, and P. Van Beek
Propositional Logic: Semantic Entailment
Learning goals
Satisfying a set of formulas
Definition of entailment
Proving/Disproving an entailment
Subtleties of an entailment
Revisiting the learning goals
Learning goals

By the end of this lecture, you should be able to:

(Semantic entailment)

- Determine if a set of formulas is satisfiable.
- Define semantic entailment.
- Explain subtleties of semantic entailment.
- Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.
Logical Deduction and Semantic Entailment

- Logic is the science of reasoning.
- The process of logical deduction is formalized by the notion of semantic entailment.
- Is a conclusion true based on a set of premises that we assume to be true?
Satisfaction of a Set of Formulas

Let $\Sigma$ be a set of premises. Let $\varphi$ be the conclusion.

A truth valuation $t$ satisfies $\Sigma$ (denoted $\Sigma^t = T$):

For any formula $\alpha$, if $\alpha \in \Sigma$, then $\alpha^t = T$. 
CQ 28 and 29 Is Sigma Satisfiable?
Semantic Entailment

Let $\Sigma$ be a set of premises. Let $\varphi$ be the conclusion.

$\Sigma$ (semantically) entails $\varphi$ (denoted $\Sigma \models \varphi$):

For any truth valuation $t$, if $\Sigma^t = T$, then $\varphi^t = T$.

$\Sigma$ does not entail $\varphi$ (denoted $\Sigma \not\models \varphi$):

There exists a truth valuation $t$ such that $\Sigma^t = T$ and $\varphi^t = F$. 
Consider the entailment \( \Sigma \models \varphi \).
To prove that the entailment holds, we need to consider

(A) Every truth valuation \( t \) under which \( \Sigma^t = T \).
(B) Every truth valuation \( t \) under which \( \Sigma^t = F \).
(C) One truth valuation \( t \) under which \( \Sigma^t = T \).
(D) One truth valuation \( t \) under which \( \Sigma^t = F \).
Disprove an entailment

Consider the entailment $\Sigma \models \varphi$.
To prove that the entailment does NOT hold, we need to consider

(A) Every truth valuation $t$ under which $\Sigma^t = T$ and $\varphi = T$.
(B) Every truth valuation $t$ under which $\Sigma^t = T$ and $\varphi = F$.
(C) One truth valuation $t$ under which $\Sigma^t = T$ and $\varphi = T$.
(D) One truth valuation $t$ under which $\Sigma^t = T$ and $\varphi = F$. 
Proving/disproving an entailment using a truth table

Let $\Sigma = \{ \neg(p \land q), (p \rightarrow q) \}, x = \neg p$, and $y = (p \leftrightarrow q)$. Based on the truth table, which of the following statements is true?

A) $\Sigma \models x$ and $\Sigma \models y$.

B) $\Sigma \models x$ and $\Sigma \not\models y$.

C) $\Sigma \not\models x$ and $\Sigma \models y$.

D) $\Sigma \not\models x$ and $\Sigma \not\models y$.

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</table>
Exercise. Show that \( \{ \neg(p \land q), (p \rightarrow q) \} \models (\neg p) \).

Exercise. Show that \( \{ \neg(p \land q), (p \rightarrow q) \} \not\models (p \leftrightarrow q) \).
Subtleties of an entailment

Consider the entailment $\Sigma \models \varphi$.
Does the entailment hold under each of the following conditions?

1. $\Sigma$ is the empty set.
2. $\Sigma$ is not satisfiable.
3. $\varphi$ is a tautology.
4. $\varphi$ is a contradiction.
CQ 34-37 Subtleties of an entailment
Revisiting the learning goals

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