Propositional Logic: Semantic Entailment

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Lecture 6

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Outline

Learning goals

By the end of this lecture, you should be able to

(Semantic entailment)

- Determine if a set of formulas is satisfiable.
- Define semantic entailment.
- Explain subtleties of semantic entailment.
- Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.

Logical Deduction and Semantic Entailment

- Logic is the science of reasoning.
- The process of logical deduction is formalized by the notion of semantic entailment.
- Is a conclusion true based on a set of premises that we assume to be true?

Satisfaction of a Set of Formulas

Let Σ be a set of premises. Let φ be the conclusion.

A truth valuation t satisfies Σ (denoted $\Sigma^t = \mathrm{T}$):

For any formula α, if $\alpha \in \Sigma$, then $\alpha^t = T$.

CQ 28 and 29 Is Sigma Satisfiable?

Semantic Entailment

Let Σ be a set of premises. Let φ be the conclusion.

Σ (semantically) entails φ (denoted $\Sigma \models \varphi$):

> *For any truth valuation t, if $\Sigma^t = T$, then $\varphi^t = T$.*

Σ does not entail φ (denoted $\Sigma \not\models \varphi$):

> *There exists a truth valuation t such that*
> *$\Sigma^t = T$ and $\varphi^t = F$.*

Prove an entailment

Consider the entailment $\Sigma \models \varphi$.
To prove that the entailment holds, we need to consider

(A) Every truth valuation t under which $\Sigma^t = \mathrm{T}$.

(B) Every truth valuation t under which $\Sigma^t = \mathrm{F}$.

(C) One truth valuation t under which $\Sigma^t = \mathrm{T}$.

(D) One truth valuation t under which $\Sigma^t = \mathrm{F}$.

Disprove an entailment

Consider the entailment $\Sigma \models \varphi$.
To prove that the entailment does NOT hold, we need to consider

(A) Every truth valuation t under which $\Sigma^t = \mathrm{T}$ and $\varphi = \mathrm{T}$.

(B) Every truth valuation t under which $\Sigma^t = \mathrm{T}$ and $\varphi = \mathrm{F}$.

(C) One truth valuation t under which $\Sigma^t = \mathrm{T}$ and $\varphi = \mathrm{T}$.

(D) One truth valuation t under which $\Sigma^t = \mathrm{T}$ and $\varphi = \mathrm{F}$.

Proving/disproving an entailment using a truth table

Let $\Sigma = \{(\neg(p \land q)), (p \to q)\}$, $x = (\neg p)$, and $y = (p \leftrightarrow q)$. Based on the truth table, which of the following statements is true?

A) $\Sigma \models x$ and $\Sigma \models y$.

B) $\Sigma \models x$ and $\Sigma \not\models y$.

C) $\Sigma \not\models x$ and $\Sigma \models y$.

D) $\Sigma \not\models x$ and $\Sigma \not\models y$.

p	q	$(\neg(p \land q))$	$(p \to q)$	$(\neg p)$	$(p \leftrightarrow q)$
F	F	T	T	T	T
F	T	T	T	T	F
T	F	T	F	F	F
T	T	F	T	F	T

Proving/disproving an entailment using the definition

Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \models (\neg p)$.
Exercise. Show that $\{(\neg(p \wedge q)), (p \rightarrow q)\} \not\models (p \leftrightarrow q)$.

Subtleties of an entailment

Consider the entailment $\Sigma \models \varphi$.
Does the entailment hold under each of the following conditions?

1. Σ is the empty set.
2. Σ is not satisfiable.
3. φ is a tautology.
4. φ is a contradiction.

CQ 34-37 Subtleties of an entailment