

# Propositional Logic: Semantic Entailment

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Lecture 6

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# Outline

## Propositional Logic: Semantic Entailment

- Learning goals

- Satisfying a set of formulas

- Definition of entailment

- Proving/Disproving an entailment

- Subtleties of an entailment

- Revisiting the learning goals

# Learning goals

By the end of this lecture, you should be able to

(Semantic entailment)

- ▶ Determine if a set of formulas is satisfiable.
- ▶ Define semantic entailment.
- ▶ Explain subtleties of semantic entailment.
- ▶ Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.

# Logical Deduction and Semantic Entailment

- ▶ Logic is the science of reasoning.
- ▶ The process of logical deduction is formalized by the notion of semantic entailment.
- ▶ Is a conclusion true based on a set of premises that we assume to be true?

# Satisfaction of a Set of Formulas

Let  $\Sigma$  be a set of premises. Let  $\varphi$  be the conclusion.

A truth valuation  $t$  satisfies  $\Sigma$  (denoted  $\Sigma^t = \top$ ):

*For any formula  $\alpha$ , if  $\alpha \in \Sigma$ , then  $\alpha^t = \top$ .*

## CQ 28 and 29 Is Sigma Satisfiable?

# Semantic Entailment

Let  $\Sigma$  be a set of premises. Let  $\varphi$  be the conclusion.

$\Sigma$  (semantically) entails  $\varphi$  (denoted  $\Sigma \models \varphi$ ):

*For any truth valuation  $t$ , if  $\Sigma^t = T$ , then  $\varphi^t = T$ .*

$\Sigma$  does not entail  $\varphi$  (denoted  $\Sigma \not\models \varphi$ ):

*There exists a truth valuation  $t$  such that  $\Sigma^t = T$  and  $\varphi^t = F$ .*

## Prove an entailment

Consider the entailment  $\Sigma \models \varphi$ .

To prove that the entailment holds, we need to consider

- (A) Every truth valuation  $t$  under which  $\Sigma^t = \text{T}$ .
- (B) Every truth valuation  $t$  under which  $\Sigma^t = \text{F}$ .
- (C) One truth valuation  $t$  under which  $\Sigma^t = \text{T}$ .
- (D) One truth valuation  $t$  under which  $\Sigma^t = \text{F}$ .



## Disprove an entailment

Consider the entailment  $\Sigma \models \varphi$ .

To prove that the entailment does NOT hold, we need to consider

- (A) Every truth valuation  $t$  under which  $\Sigma^t = \text{T}$  and  $\varphi = \text{T}$ .
- (B) Every truth valuation  $t$  under which  $\Sigma^t = \text{T}$  and  $\varphi = \text{F}$ .
- (C) One truth valuation  $t$  under which  $\Sigma^t = \text{T}$  and  $\varphi = \text{T}$ .
- (D) One truth valuation  $t$  under which  $\Sigma^t = \text{T}$  and  $\varphi = \text{F}$ .

## Proving/disproving an entailment using a truth table

Let  $\Sigma = \{(\neg(p \wedge q)), (p \rightarrow q)\}$ ,  $x = (\neg p)$ , and  $y = (p \leftrightarrow q)$ . Based on the truth table, which of the following statements is true?

- A)  $\Sigma \models x$  and  $\Sigma \models y$ .
- B)  $\Sigma \models x$  and  $\Sigma \not\models y$ .
- C)  $\Sigma \not\models x$  and  $\Sigma \models y$ .
- D)  $\Sigma \not\models x$  and  $\Sigma \not\models y$ .

$p$	$q$	$(\neg(p \wedge q))$	$(p \rightarrow q)$	$(\neg p)$	$(p \leftrightarrow q)$
F	F	T	T	T	T
F	T	T	T	T	F
T	F	T	F	F	F
T	T	F	T	F	T

## Proving/disproving an entailment using the definition

Exercise. Show that  $\{(\neg(p \wedge q)), (p \rightarrow q)\} \models (\neg p)$ .

Exercise. Show that  $\{(\neg(p \wedge q)), (p \rightarrow q)\} \not\models (p \leftrightarrow q)$ .

## Subtleties of an entailment

Consider the entailment  $\Sigma \models \varphi$ .

Does the entailment hold under each of the following conditions?

1.  $\Sigma$  is the empty set.
2.  $\Sigma$  is not satisfiable.
3.  $\varphi$  is a tautology.
4.  $\varphi$  is a contradiction.

## CQ 34-37 Subtleties of an entailment

# Revisiting the learning goals

By the end of this lecture, you should be able to

(Semantic entailment)

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- ▶ Prove that a semantic entailment holds/does not hold by using the definition of semantic entailment, and/or truth tables.